Reading 1

Sets and introductions

Let's first review some common conventions about how we talk about *sets* as mathematicians. The review will not be extensive, as you are assumed (at this university) to have already taken a class on basics of sets and proof.

1.1 Sets and elements

Recall that a *set* is a collection of objects. If A is a set, an *element of A* is an object in A.

Notation 1.1.1. If (and only if) a is an element of A, we write $a \in A$. If multiple elements a, a', and a'' are in A, we may write $a, a', a'' \in A$.

We may sometimes say that "a is in A" or "a is contained in A" as well.

Example 1.1.2. Let $\mathbb{Z}$ be the set of all integers. Then the integer 3 is an element of $\mathbb{Z}$. We write $3 \in \mathbb{Z}$. Other examples of elements of $\mathbb{Z}$ include -2, 0, and $10,000,000,009$.

Let $\mathbb{Q}$ be the set of all rational numbers. (Numbers that can be expressed as fractions of integers.) Then $-1/3 \in \mathbb{Q}$, and $2/5 \in \mathbb{Q}$. Note that any integer is also a rational number. For example, $3 = 3/1$.

Let $\mathbb{R}$ be the set of all real numbers. Then $\pi, -\sqrt{2}, e \in \mathbb{R}$. Recall that any rational number is a real number.

Remark 1.1.3. It is very common to denote sets by capital letters, and to denote elements of that set by the same letter, but lower-case. For example, you will often read: "Let A be a set, and fix $a \in A$."

While this convention is not always followed, I would encourage it. The major exception to this convention is when sets are “famous,” or already come with a set notation, or have a different font. For example, $\mathbb{R}$ is the set of real numbers, but we more commonly write x, y, t for real numbers than r. Likewise, we rarely write z for an integer; instead using letters like a, b, m, n, i, j.

Warning 1.1.4. We have not defined (in this course) what a *set* actually is. This is very tricky business, and would take a long time to set up sets accurately and precisely. See Exercises 1.7.1 and 1.7.2. In this course, we bring your attention to this danger as a boogeyman, then hereafter ignore it.

Notation 1.1.5. We will write

$$A = \{5, 7, 8\}$$

to mean that A is a set consisting of exactly three elements, called 5, 7, and 8. Likewise, the notation

$$B = \{\text{Bob, banana, Alice, } 1\}$$

means that B is a set consisting of exactly four elements, called Alice, Bob, 1, and banana.

1.2 Subsets

Let B be a set. We say that A is a *subset of* B if (and only if) every element of A is an element of B.

If A is a subset of B, we will write

$$A \subset B.$$

Example 1.2.1. Let $A = \{1, 3, 5\}$. Then A is a subset of $\mathbb{Z}$.

Intuition 1.2.2. Let B be a set. You might imagine that B is a bag containing things, and an *element* of B is simply a thing in the bag.

Now imagine dumping the contents of B into a water tank. Take a big ladle, or some sort of scoop, and take a big scoop. Whatever you collect in your ladle is an example of a subset of B.

1.3 The empty set

The *empty set* is the set containing no elements. We denote the empty set by the symbol

$$\emptyset.$$

Intuition 1.3.1. Think back to Intuition 1.2.2. Note that, when you scoop, you might come out empty-handed (you might scoop up nothing)! This is an example of the empty set.

This thought experiment/intuition is supposed to reinforce the idea that *the empty set is a subset of any set.* Put another way, for any set B, we may write $\emptyset \subset B$.

1.4 Proving two sets are equal

We say that two sets A and B are *equal* if they have exactly the same elements, and we will write

$$A = B$$

when two sets are equal.

The most common way to prove that two sets A and B are equal is to show that $A \subset B$ *and* $B \subset A$.

Conversely, if $A = B$, then $A \subset B$ and $B \subset A$.

1.5 Sets and whole numbers

A lot of students get confused about the relationship between sets and numbers.

Given any set, we can ask "how many elements are in the set?" We can ask it, but we may not always get an answer we're familiar with. This is because "how many" is a question that is usually answered by a whole number, like 0, 1, 2, 3, et cetera. If you have taken a course where you learned about the term *cardinality*, you know that there are even bigger kinds of numbers, like different kinds of infinity, which cannot be expressed by a whole number.

For example, "how many elements are in the set $\mathbb{Z}$" and "how many elements are in the set $\mathbb{R}$" are questions that turn out to have *different* answers, because while both sets are infinitely large, one can prove that the cardinality of $\mathbb{R}$ is strictly larger than the cardinality of $\mathbb{Z}$.

Of course, two different sets may have the exact same number of elements, but the sets may be unequal. For example, if A is a set containing three apples, and B is a set containing three bananas, then A and B have the same cardinality, but they are clearly different sets.

1.6 In-class exercises: The power set

Let A be a set. The *power set* of A is the set

$$\mathcal{P}(A) := \{B \subset A\}.$$

That is, $\mathcal{P}(A)$ is the set of all subsets of A.

Exercise 1.6.1.

1. Let A_0 be the empty set. How many elements are in A_0?
2. Write out all the elements of $\mathcal{P}(A_0)$.
3. How many elements are in $\mathcal{P}(A_0)$?

Exercise 1.6.2.

1. Let A_1 be the set containing a single element called a.
2. Write out all the elements of $\mathcal{P}(A_1)$.
3. How many elements are in $\mathcal{P}(A_1)$?

Exercise 1.6.3.

1. Let A_2 be the set containing exactly two elements, called a and b.
2. Write out all the elements of $\mathcal{P}(A_2)$.
3. How many elements are in $\mathcal{P}(A_2)$?
4. What does $\mathcal{P}(A_2)$ have to do with squares?

Exercise 1.6.4. Let A_3 be the set containing exactly three elements, which we call a, b, and c. What does $\mathcal{P}(A_3)$ have to do with cubes?

Exercise 1.6.5. Fix an integer $n \geq 0$. Let A be a set containing exactly n elements. How many elements do you think are in $\mathcal{P}(A)$? Can you prove it?

If a square is a "two-dimensional" analogue of a cube, and a cube is a "three-dimensional" analogue of a cube, do you have any ideas as to how you might draw the four-dimensional analogue of a cube?

1.7 Exploratory exercises (optional)

Exercise 1.7.1. In this exercise, you will study the notion of "a set of all sets." Your study will give strong evidence (to you) that we must be careful what we mean by a set. In this course, we will introduce this danger as a boogeyman, then hereafter ignore it.

Exercise: Study the following proof. What, if anything, is wrong with it?

Let X be the set of all sets, and let $\mathcal{P}(X)$ be the power set. By definition of X, X must contain $\mathcal{P}(X)$. But by Cantor's Theorem, the power set of any set has strictly higher cardinality than the original set—so X could not contain $\mathcal{P}(X)$.

We have thus begun with the assumption that the set of all sets exists, and ended with a contradiction.

Exercise 1.7.2. Here is another exercise showing how careful we must be when defining sets.

Exercise: Study the following proof. What, if anything, is wrong with it?

Let X be the set of all sets that do not contain themselves as an element. For example, the set $A = \{2, 3\}$ does not contain itself.[1] We may ask whether X is in itself. If $X \in X$, we arrive at a contradiction due to the definition of X. If $X \notin X$, then by definition of X, we again arrive at a contradiction.

1.7.1 Functions

A *function* from set A to a set B is a way to assign an element of B to every element of A. We will often name functions with letters like f, g, q, p, ϕ, ψ. (These last two are the Greek letters pronounced "phi" and "psi".)

[1] You may now wonder—is there a set that contains itself? This question gets to the heart of the "foundations" of mathematics. The most commonly used axioms of set theory exclude such a possibility, in that one can prove from the axioms that no set can contain itself. However, there are other axioms of set theory that do allow for sets that contain themselves.

Notation 1.7.3. The string of symbols

$$f : A \to B$$

means that f is a function from A to B. We say that A is the *domain* of f, and that B is the *codomain* of f.

The string of symbols

$$a \mapsto f(a)$$

means that the function f assigns the element $a \in A$ to the element $f(a) \in B$.

Note that there are two kinds of arrows in the lines above: $\to$, which helps tell us what the domain and codomain are, and $\mapsto$, which tells us what f actually *does* on elements.

Example 1.7.4. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2 + 3.$$

This is the function from the real line to itself that sends a real number x to the real number obtained by squaring x and then adding 3. For example, $f(2) = 7$.

Notation 1.7.5. We will often say in words "let f be a function from A to B." In this class, I will often be lazy and write "let $f : A \to B$" for the exact same meaning.

Remark 1.7.6. As you know, in mathematical writing, the phrase "Let ... be a *blah*" means that whatever we say next has to be true no matter which *blah* we choose! So in such situations, we will typically *not* have any formula that describes f except for the tautological statement that f sends a to $f(a)$.

This is often surprising to students who are coming straight from calculus—functions may be described not by formulas. In this class, they will often be described using words. Functions may, in fact, be not described at all except for the knowledge that they are functions! (For example, "let f be a function from A to B" says nothing about what f actually does.)

In other words, we will be able to say things and conclude things about functions without knowing very much about them. This would be rude if such functions were people, but at least the things we say will be true.

Remark 1.7.7. For some students, it is mind-boggling to make statements about a function without knowing what the function is. But such abstraction is something you are already used to. For example, if I tell you to tell me things about a left hand, but I do not tell you *whose* left hand I'm talking about, you could still tell me things about a left hand (because you know that some facts are true of all left hands, regardless of who owns the left hands). We will be making similar deductions about arbitrary functions.

Example 1.7.8 (The identity function)**.** Every set A admits a very special function to itself:

$$\mathrm{id} : A \to A, \qquad a \mapsto a.$$

This function sends an element a to itself. It is one of the few families of functions for which we can write a "formula," namely the formula $f(a) = a$.

This is called the *identity function of A.* Note that we don't use the symbol f above; this function is special enough that it gets its own notation: id.

Example 1.7.9 (Constant functions)**.** Let A and B be sets. Suppose B is not the empty set, and choose an element $b \in B$ once and for all. Then the *constant function* with image b is the function

$$A \to B, \qquad a \mapsto b$$

sending every element of A to b. You can see why it's called constant—changing a doesn't change $f(a)$. This is another example of one of the few families of functions for which we can write a "formula," namely the formula $f(a) = b$.

Note that, if A is also non-empty, there are as many constant functions from A to B as there are elements of B.

Example 1.7.10 (Inclusion functions)**.** Let S be a subset of A. Then there is another special function called the *inclusion function* (of S into A)

$$\iota : S \to A, \qquad s \mapsto s.$$

The symbol ι is the Greek letter "iota," though you will not be punished for writing the letter i instead. ι has domain S and codomain A. Confusingly, or naturally, an element $s \in S$ is sent to "itself," but considered as an element of A. This is another example of one of the few families of functions for which we can write a "formula," namely the formula $\iota(s) = s$.

The inclusion function is *not* the identity function unless we choose our subset S to equal A itself. Note that the formulas for ι and id are identical, even though the functions are different (when $S \neq A$)! The reasons the functions are different is because their domains are different. This is another reason it is important to remember that we must always specify the domain and codomain of a function.

1.8 Images

Definition 1.8.1 (Image)**.** Let $f : A \to B$ be a function, and choose a subset $S \subset A$. The *image of* S *under* f (or "the image of S" for short) is the set of all elements in B that are "hit" by elements of S. More precisely, the image of S under f is the set

$$\{b \in B \,|\, \text{There exists some } a \in S \text{ for which } f(a) = b.\}$$

When we choose our subset S to be A itself, we will simply say "the image of f."

Notation 1.8.2. We will often denote the image of S by the symbol $f(S)$.

Remark 1.8.3. Note that $f(S) \subset B$.

Warning 1.8.4. Let $f : A \to B$, fix a subset $S \subset A$, and let $s \in S$. The notations $f(S)$ and $f(s)$ mean very different things. $f(S)$ is the image of S under f, and it is a subset of the codomain. $f(s)$ is the result of applying f to a single element s, and $f(s)$ is an *element* of B.

Note that even if S is the subset $\{s\}$ consisting of a single element, we still have that $f(\{s\})$ is not the same thing as $f(s)$. The former is a subset of B (consisting of the single element $f(s)$), while the latter is an element of B.

Example 1.8.5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $x \mapsto x^2 + 3$. Then the image of f is the set of all real numbers that are bigger than or equal to 3.

Example 1.8.6. Fix a set A and a subset $S \subset A$. Let $\iota : S \to A$ be the inclusion function (Example 1.7.10). Then the image of ι is precisely S itself.

Here is a definition you have seen before:

Definition 1.8.7. Let $f : A \to B$ be a function. We say f is a *surjection*, or we say f is *onto*, if $f(A) = B$.

1.9 Pre-images

Definition 1.9.1 (Pre-image)**.** Let $f : A \to B$ be a function, and choose a subset $T \subset B$. The *pre-image of* T *under* f (or "the pre-image of T" for short) is the set of all elements in A that are sent to elements of T. More precisely, the pre-image is the set

$$\{a \in A \mid f(a) \in T\}.$$

Notation 1.9.2. We will often denote the pre-image of T by the symbol $f^{-1}(T)$.

Remark 1.9.3. Note that $f^{-1}(T) \subset A$.

Warning 1.9.4. The symbol f^{-1} here does *not* mean the "inverse function to f." For example, if f is not a bijection, there may not even be an inverse function! This is another piece of evidence that the superscript $^{-1}$ is overused in mathematics. Here, the meaning of the symbol f^{-1} does not become clear until we realize that it is being applied to a subset of the codomain—that is, until we see the whole expression $f^{-1}(T)$. In contrast, it does not make sense to write $f^{-1}(b)$ for some element $b \in B$, as f may not have an inverse function. It *does* make sense to write $f^{-1}(\{b\})$.

Example 1.9.5. Let $f : A \to B$ be any function. Then $f^{-1}(B) = A$ and $f^{-1}(\emptyset) = \emptyset$.

Example 1.9.6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $x \mapsto x^2 + 3$. Then the pre-image of the set $\{12\} \subset \mathbb{R}$ is the set $\{-3, 3\}$. The pre-image of the set $[12, \infty)$ is the set

$$(-\infty, -3] \bigcup [3, \infty).$$

Example 1.9.7. Fix a set A and a subset $S \subset A$. Let $\iota : S \to A$ be the inclusion function (Example 1.7.10). Then the pre-image of $T \subset A$ is precisely the intersection $S \cap T$ (considered as a subset of S).

Here is a definition you have seen before:

Definition 1.9.8. Let $f : A \to B$ be a function. We say f is an *injection* if for every $b \in B$, the pre-image $f^{-1}(\{b\})$ consists of at most one element.

Definition 1.9.9. If f is both an injection and a surjection, we say that f is a *bijection*.

Warning 1.9.10. Some textbooks use the phrase “one-to-one function” to mean an injection, and “one-to-one correspondence” to mean a bijection. I will avoid both terms.