Closures and Denseness

(extra:

$S^n \cong (\mathbb{R}^n)^+$  $\mathbb{R}^2$)

Def: fix X a space. fix $B \subset X$. the closure of B is the intersection of all closed subsets containing B

$$\text{int}(B) = \bigcup_{\substack{U \subset X \text{ open} \\ \text{and } U \subset B}} U$$

$$\overline{B} = \text{closure of } B = \bigcap_{\substack{K \subset X \text{ closed} \\ \text{and } B \subset K}} K$$

ex) $B = X$, $\overline{B} = B$

ex) $B = \emptyset$, $\overline{B} = \emptyset$

ex) suppose B is closed in X then $\overline{B} = B$

Compare) If $U \subset X$ is open, then $\text{int}(U) = U$

Proof) $\overline{B} = \bigcap_{\substack{K \text{ closed in } X \text{ and} \\ B \subset K}}$

$\Rightarrow$

note $B \subset \overline{B}$ (If $x \in B$, then $x \in K$ for all K containing B. So $x \in \bigcap K$)

to see $\overline{B} \subset B$ note that B is a closed subset containing B, so

$$\overline{B} \subset \bigcap K \subset \left(\bigcap_{K \neq B} K\right) \cap B \subset B \quad \blacksquare$$

Proposition) B is closed iff $\overline{B} = B$

$\Leftrightarrow$

Proof) $\overline{B}$ is closed bc $\overline{B}$ is an intersection of closed subsets.

$\Leftarrow$ since $B = \overline{B}$, B is closed remark: informally $\overline{B}$ is the smallest closed subset containing B

Def: Fix X a space. fix $B \subset X$. We say B is dense in X if $\overline{B} = X$

Prop) the following are equivalent:

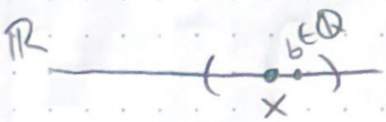
1) B is dense in X
2) $\forall x \in X$ and $\forall$ nbhd $N \ni x$ $N \cap B \neq \emptyset$
3) $\forall x \in X$ and $\forall$ open U $\underset{\text{in } X}{\wedge}$ w/ $x \in U$ $U \cap B \neq \emptyset$

(note if B is closed and dense, then $B = X$)

ex) $X = \mathbb{R}$, $B = \mathbb{Q}$

claim: $\mathbb{Q}$ is dense in $\mathbb{R}$

pf: by prop. it suffices to show that $\mathbb{Q}$ satifies 3)



choose $x \in \mathbb{R}$. WTS that for any open $U \subset \mathbb{R}$ st $x \in U$, $\exists$ some $b \in \mathbb{Q}$ in U

since U is open $\exists \varepsilon > 0$ st $(x-\varepsilon, x+\varepsilon) \subset U$. bc $\varepsilon > 0$ $\exists$ some $N \in \mathbb{Z}$ st $10^{-N} < \varepsilon$.

[ex) $\varepsilon = .0032121212121\ldots$, $10^{-3} = .001$ $\Big\}$ $10^{-3} < \varepsilon$

$\varepsilon = .000\ldots\square$ ← non zero so $\varepsilon > .000\ldots001 = 10^{-N}$]

let b be a decimal # whose first $N+1$ digits after the decimal point agree with those of x ...

cont →

... and has digits zero there after. so $b \in \mathbb{Q}$ and $|x-b| < 10^{-N}$

so $|x-b| < \varepsilon$ meaning $b \in U$

end class

extra 5 min notes)

topology helps us study shapes we cannot visualize

ex)