11/14/2023

Notes

Q: T/F If P is a poset, P (with Alexandroff topology) is hausdorff iff P is discrete.

ex: $P = [3] = \{0,1,2,3\}$ ($\leq$ relation)

$\mathcal{T} = \{\emptyset, \{3\}, \{2,3\}, \{1,2,3\}, P\}$

consider $P = \{1,2\}$ under $R := \{(1,1),(2,2)\}$

$\mathcal{T}_P = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ is both discrete and hausdorff.

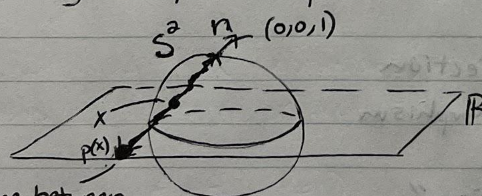
for $x = 1$, $x' = 2$ ($U = \{1\}$, $U' = \{2\}$)
U, U' are both open in P and $U \cap U' = \emptyset$

Today

One-point compactification

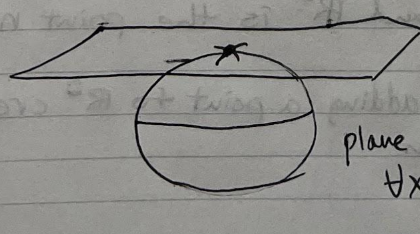
Motivating example: $p: S^2 \setminus \{\text{north pole}\} \to \mathbb{R}^2$

$S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$



$p(x) = (p(x)_1, p(x)_2, 0)$

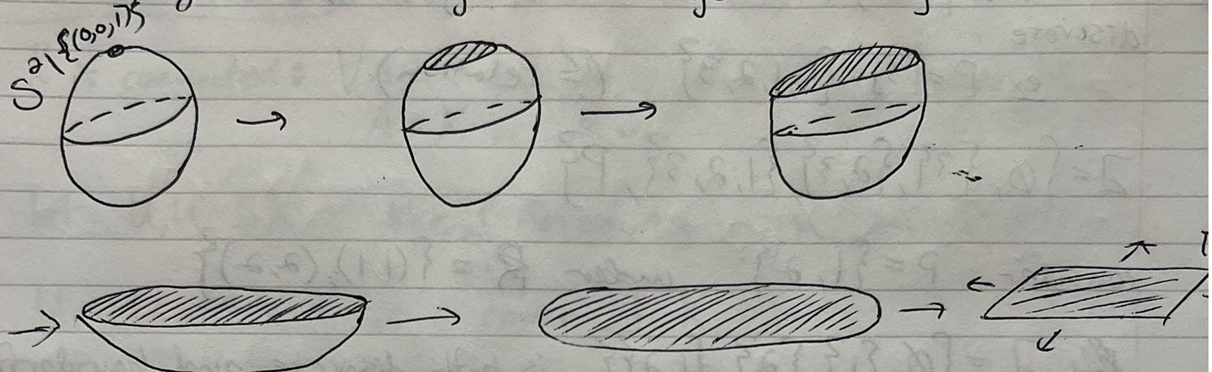
line between x and n intersects the plane $\{(x,y,0) \mid x,y \in \mathbb{R}\} \subseteq \mathbb{R}^3$



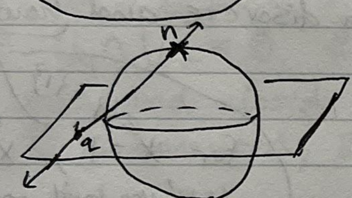
~~height of the~~ plane height can't be 1 or else $p(x) = p(y)$ $\forall x,y \in S^2$.

Is p continuous? Well-defined? bijective? homeomorphic?

yes yes yes yes



p^{-1}



for any $q \in \mathbb{R}^2$,
draw the line between q and the north pole n.

then the line intersects S^2 at exactly ~~the~~ one point.
(any line goes through $n \notin S^2 \setminus \{(0,0,1)\}$, and a line may only intersect the sphere at most 2 points. So p^{-1} is cts. bijection.

~~So p is a bijection~~

So p is cts.
p^{-1} is cts.
p is a bijection
$\therefore$ p is a homeomorphism

P is called "stereographic projection"

So the only difference between S^2 and $\mathbb{R}^2$ is the point n.

We can follow this logic backwards, adding a point to $\mathbb{R}^2$ creating S^2. Called one-point compactification.

Def. One-point Compactification

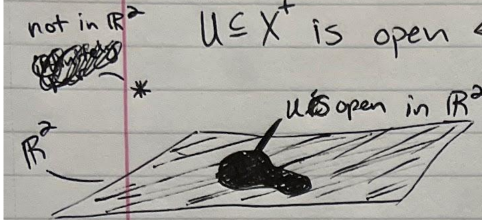
$*$ is some point (commonly "the point at infinity")

Let X be a top space. Define $X^+ := X \cup \{*\}$.

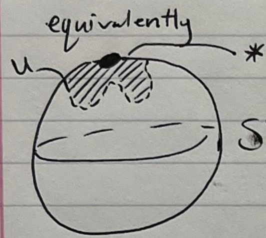
Endow X^+ with the topology:

$U \subseteq X^+$ is open $\iff$ (1) If $* \notin U$, and U is open in X

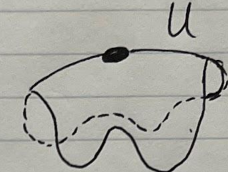
(2) If $* \in U$, and $U^c \subseteq X$ is closed and compact



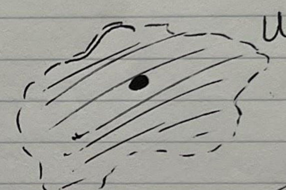
U is open in $\mathbb{R}^2 \Rightarrow U$ is open in $\mathbb{R}^{2+}$



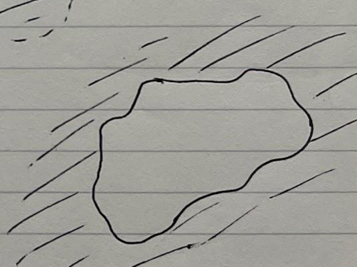
$S^2 \setminus \{(0,0,1)\} = X$



$\downarrow$ Flatten into $\mathbb{R}^2$



$U^c =$



$\therefore$ U is open.