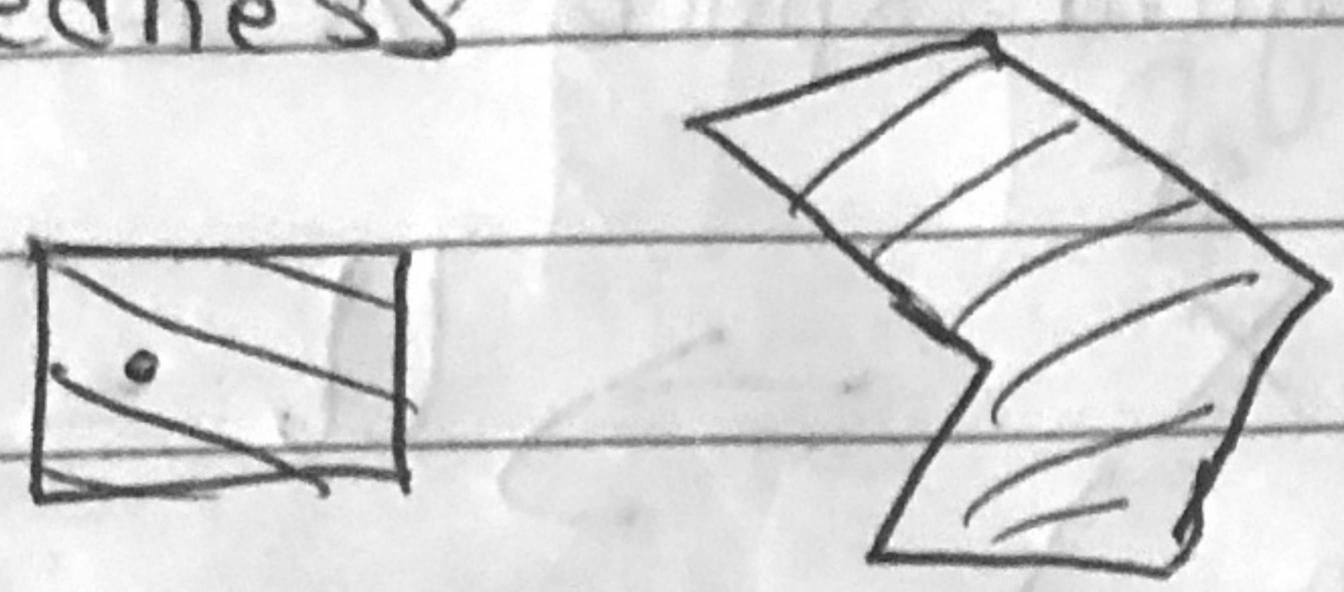
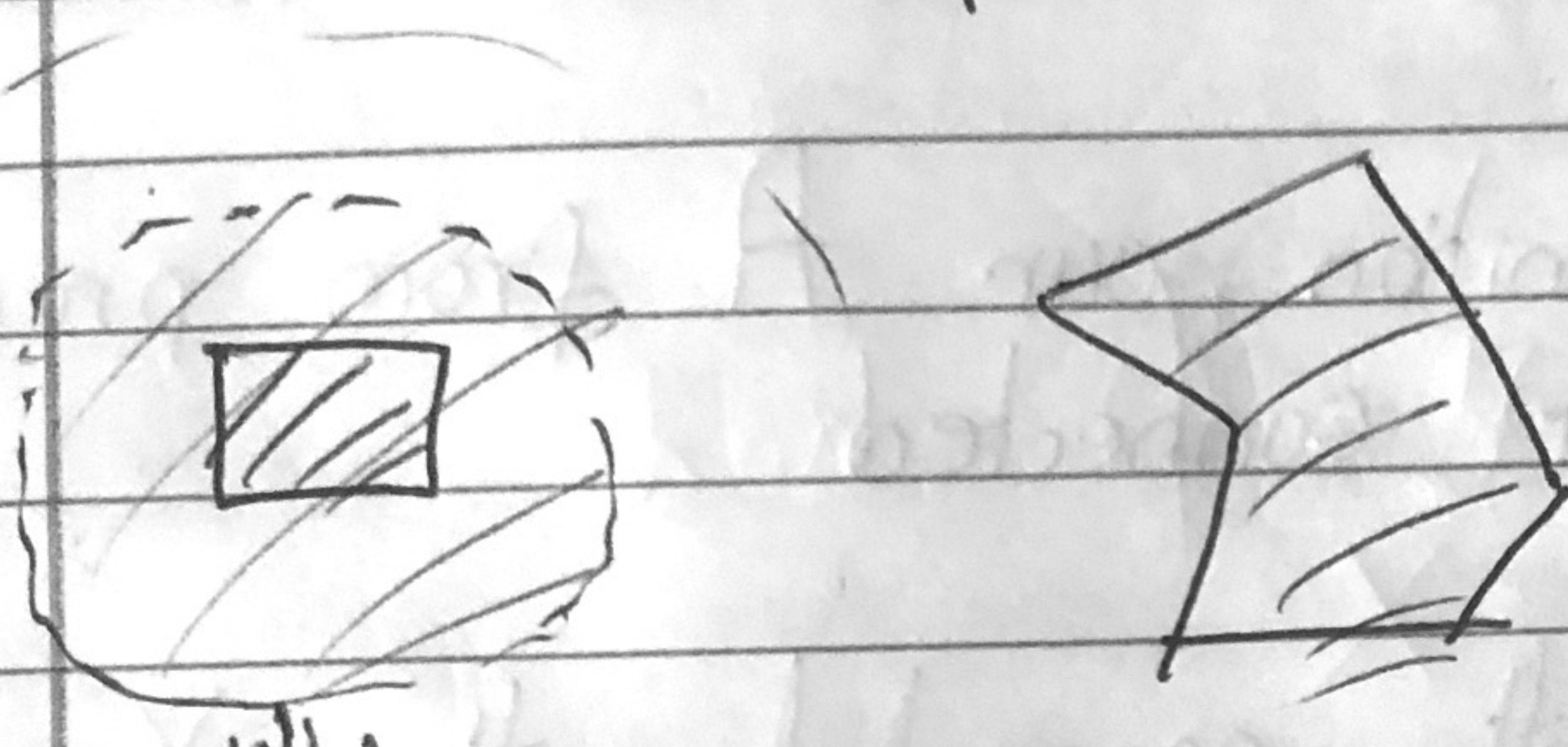
11/7/2023

Connectedness

$A =$  $\subset \mathbb{R}^2$

A is not path connected

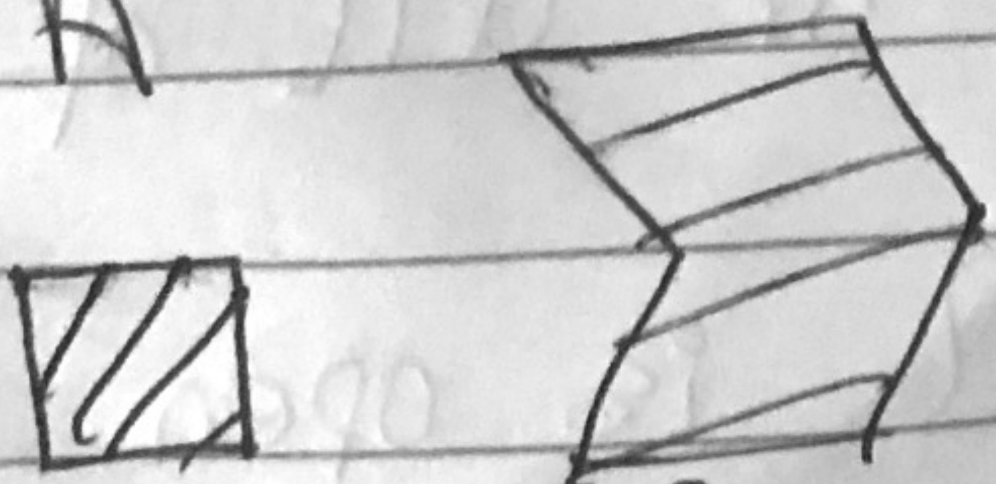
$=$ 

W

A subset $U \subset A$ is open iff $\exists$ open $W \subset \mathbb{R}^2$
such that $U = W \cap A$

The black square is open in A.
The black square is open and closed in A

* Redrawing A

$A =$ 

This shape is also open and closed.

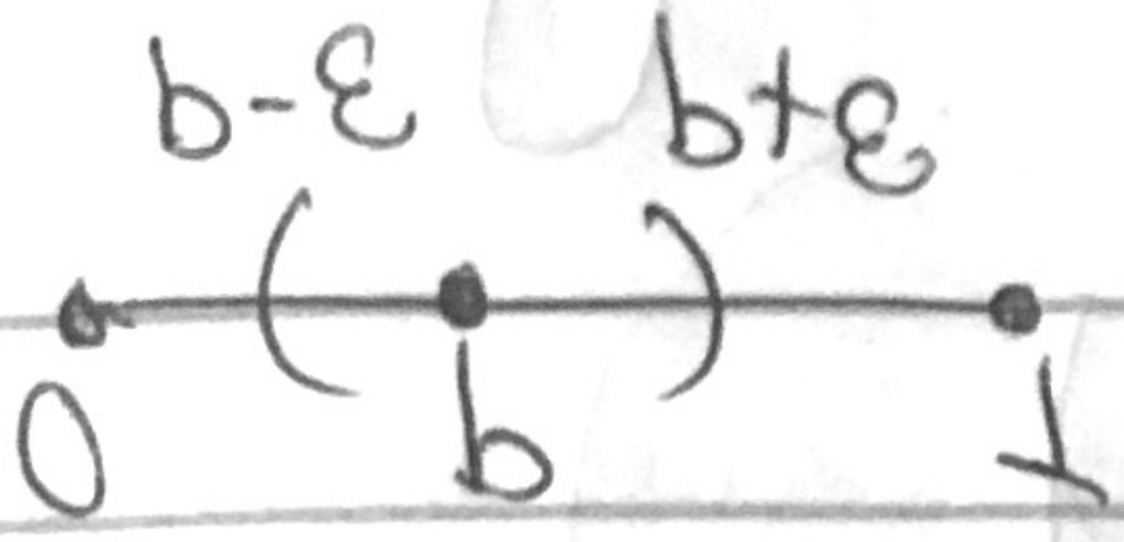
Definition: A topological space X is called connected if
$U \subset X \Rightarrow U = \phi$ or X
(Where U is open and closed)

from this definition, our A from previous figure is not connected

Proposn: If X is path connected, then X is connected.

Proof of this Proposn ~~is true if this~~
relies on $\Rightarrow$ Lemma: $[0,1]$ is connected
Proof of lemma:
Given $U \subset [0,1]$
Such that U is open and closed
U is empty or all of $[0,1]$

(I) If $U \subset [0,1]$ is open and if
$b \in U$
$b \neq 0, 1$
then $\exists\ \varepsilon > 0$ such that
$(b-\varepsilon, b+\varepsilon) \subset [0,1]$



(I) If $U \subset [0,1]$ is closed then U contains a maximal element.

U is closed in closed interval

U is closed in all interval

U is bounded

$\Rightarrow$ It is impossible for U to be closed open and not contain in $[0,1]$

$b_{max} \in U$

Here we used Heine-Borel theorem and Extreme Value theorem

* Disconnected:

A topological space is called disconnected if X is not connected.

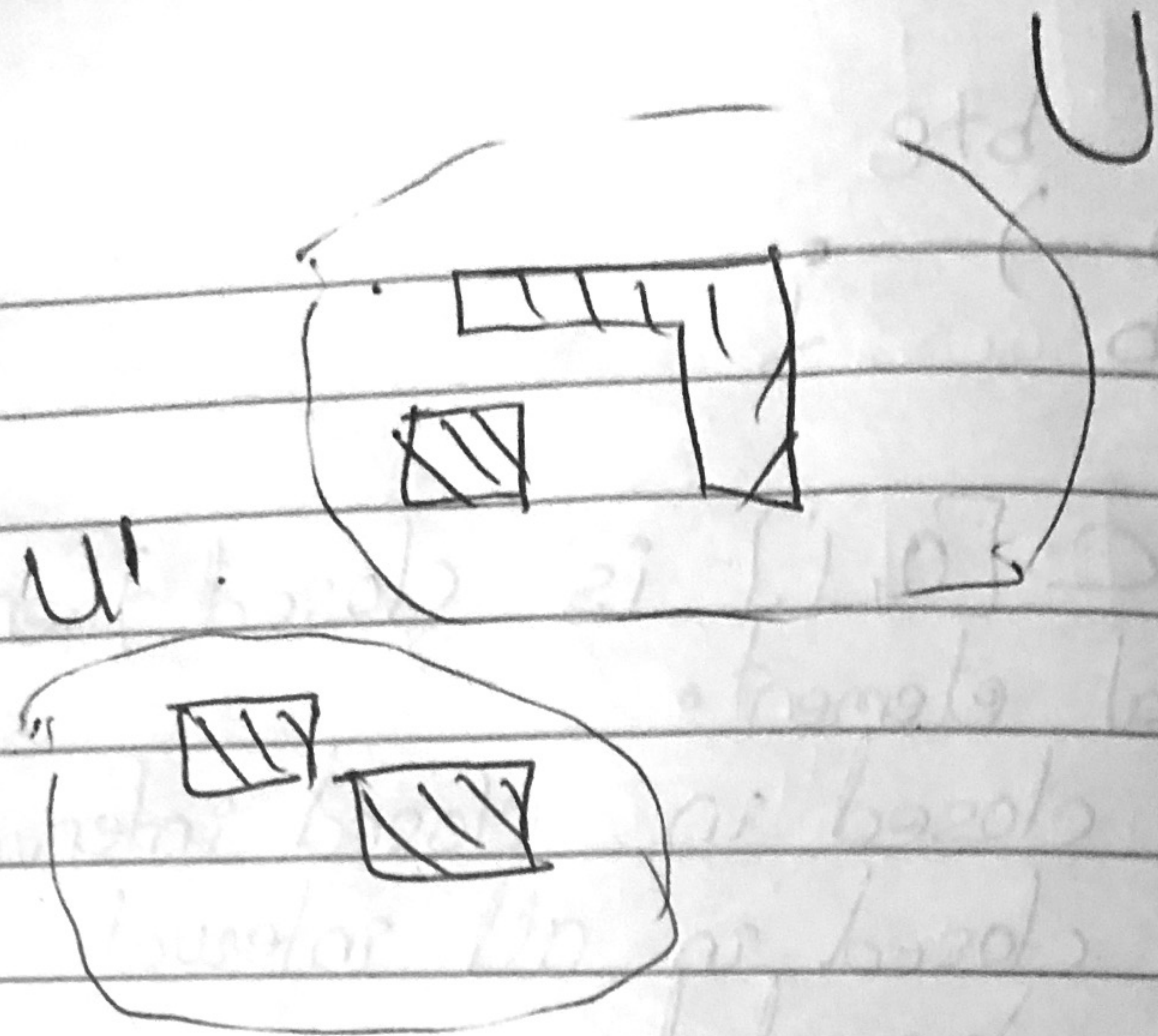
Proposition: The following are equivalent:

(1) X is disconnected

(2) $\exists$ two open ~~sets~~ subsets $U, U' \subset X$ such that $U \cup U' = X$ and

$$U \cap U' = \phi$$

* $X =$



Here $U \cap U' = \phi$ and
$U \cup U' = X$