If $m \neq n$, is $\mathbb{R}^m$ homeomorphic to $\mathbb{R}^n$?
When $m=0$, $\mathbb{R}^m = \mathbb{R}^0$ is a point so obviously no bijection to any other $\mathbb{R}^n$
When $m,n \neq 0$, $\mathbb{R}^m$ admits a bijection to $\mathbb{R}^n$
Thm (Invariance of Domain): $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ iff $m=n$.
We will use path connectedness to prove invariance of domain for $m=1$

Some observations: Fix a space X. From last time, x is connected by a path to x' iff $\exists$ continuous $\gamma : [0,1] \to X$ w/ $\gamma(0) = x$ and $\gamma(1) = x'$. Define a relation $R \subset X \times X := \{(x,x') \mid x \text{ is path connected to } x'\}$

① $\forall x \in X$, $(x,x) \in R$

Consider $\gamma : [0,1] \to X$
$t \to x$
if $U \subset X$, if $x \notin U$, then $\gamma^{-1}(U) = \emptyset$.
If $x \in U$, $\gamma^{-1}(U) = [0,1]$
Both $\emptyset, [0,1]$ open so γ is continuous

② $\forall x,x' \in X$, if $(x,x') \in R$, then $(x',x) \in R$
- if given $\gamma : [0,1] \to X$
consider $[0,1] \xrightarrow{j} [0,1] \xrightarrow{\gamma} X$
when j is homeomorphism sending $0 \mapsto 1$, $1 \mapsto 0$

③ $\forall x,x',x'' \in X$, if $(x,x') \in R$ and $(x',x'') \in R$, then $(x,x'') \in R$.

So given γ, a path from x to x', and γ', a path from x' to x'', we can define $\gamma' \# \gamma : [0,1] \to X$ by
$$t \longmapsto \begin{cases} \gamma(2t) & \text{when } t \in [0, \frac{1}{2}] \\ \gamma(2t-1) & \text{when } t \in [\frac{1}{2}, 1] \end{cases}$$

Proposition: R is an equivalence relation

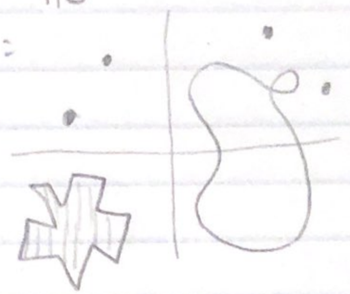
Defn: Fix a topological space X.

$\pi_0 = X/x \sim x'$ iff $\exists$ a continuous path from x to x'

(also called the set of path connected components of X)

ex. $X \subset \mathbb{R}^2$

X =



In this example, $\pi_0(X)$ has 6 elements (6 eq classes)

Proposition: If $f: X \to Y$ is a homeomorphism then f induses a bijection $\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$