So far: Defined "topological space": $(\underset{\text{set}}{X}, \underset{\text{A topology}}{\mathcal{T}})$

(I) Producing Examples:

- $\mathbb{R}^n$ Euc. space
- P Posets
- $X \times Y$ Products
- $X/\sim$ Quotient
- $A \subset X$ Subspaces
- (X, d) Metric spaces

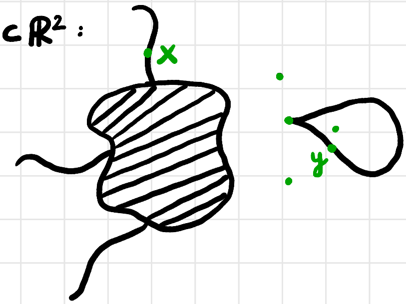
(A) Find ways to describe spaces

- compact (Thm: [Heine-Borel] $A \subset \mathbb{R}^n$ compact $\Leftrightarrow$ A is closed & bounded)
- Hausdorff (Thm: X is hausdorff $\Leftrightarrow$ $\Delta \subset X \times X$ is closed)

Properties of Spaces satisfy:

- If X is homeomorphic to Y, then X has property $P \Leftrightarrow Y$ has property P

$A \subset \mathbb{R}^2$:



Not path-connected

Def: Let X be a topological space. We say X is path-connected if $\forall\, x, x' \in X$, there exists a path connecting x to x'

Defn: A path in a top. space, X, is a cts. fxn $\gamma : [0,1] \to X$
(w/ the subspace topology inherited from $(\mathbb{R}, \mathcal{T}_{std})$)

We say γ is a path from $\gamma(0)$ to $\gamma(1)$ & that $\gamma(0)$ connects to $\gamma(1)$

Ex: $\mathbb{R}$ is path-connected

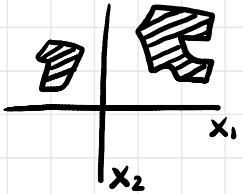
Pf: Fix $x, x' \in \mathbb{R}$. We WTS $\exists$ cts. fxn $\gamma : [0,1] \to \mathbb{R}$ s.t. $\gamma(0) = x$ & $\gamma(1) = x'$. Consider the function

$$[0,1] \to \mathbb{R}$$
$$t \longmapsto x + t(x' - x)$$

Note: $\gamma(0) = x = 3$ $\quad \gamma(0) = x$ ✓
$\gamma(1) = x' = -9$ $\quad \gamma(1) = x'$ ✓

γ is cts. (You can use ε-δ or appeal to Hiro's claim)

Ex: Let

$A =$  $\subset \mathbb{R}^2$ Then A is not path connected

Pf: WTS $\exists\, x, x' \in A$ s.t. $\forall \gamma : [0,1] \longrightarrow A$ w/ $\gamma(0) = x$, $\gamma(1) = x'$. γ cannot be cts. So choose $x \in A$ s.t. $x_1 < 0$, $x' \in A$ s.t. $x'_1 > 0$.
If $\gamma(0) = x$, & $\gamma(1) = x'$, then $\gamma(0)_1 < 0$ & $\gamma(1)_1 > 0$.

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\gamma"] \arrow[dr] & A \subset \mathbb{R}^2 \arrow[d, "P_1"] \\
& \mathbb{R}
\end{tikzcd}$$

Because composition preserves continuity, "γ cts $\Rightarrow \gamma_1 = P_1 \circ \gamma$ is cts"
But $\nexists$ element of A w/ x-coordinate equal to zero

(1) Pictures v. Existence of fxns
(2) How're we using IVT?
(3) Connected v. Path-connected