10/24/23
true

Contrapositive of $p \Rightarrow q$ is "$\neg q \Rightarrow \neg p$"

"not $q \Rightarrow$ not p"

(6) "not hausdorff $\Rightarrow$ Δ not closed and Δ not open"

So contrapositive is "if Δ is closed or open then X is hausdorff."

They're both equivalenty logical.

Today: Metric Spaces

Informally: a metric space is a set X equipped w/ a way to measure "the" distance b/t any 2 points of X.

(To be able) to give a distance to any pair of points is to give a function $d: X \times X \longrightarrow \mathbb{R}$, $(x,x') \longmapsto d(x,x')$

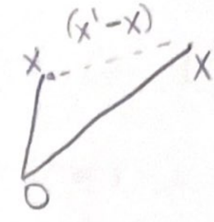
satisfying: (1) $\forall x \in X$, $d(x,x') = 0 \Rightarrow x = x'$ (non-degeneracy)

(2) $\forall x, x' \in X$, $d(x,x') = d(x',x)$ (symmetry)

(3) $\forall x, x', x'' \in X$, $d(x,x') \leq d(x,x') + d(x',x'')$ (triangle inequality

$\longrightarrow \leq \text{---)} + \text{---)}$

definition a metric space is a set X equipped w/ a metric d. (a metric is a function). We often say that (X, d) is a metric space or that X is a metric space (leaving d implicit)



Ex Let $X = \mathbb{R}^n$, and $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$(x, x') \mapsto \sqrt{\sum_{i=1}^{n} (x_i' - x_i)^2}$$

This is called the standard metric.

Proof (of (1)): clearly $d(x, x) = 0$

on the other hand, if $d(x, x') = 0$, then $\sum_{i=1}^{n} (x_i' - x_i)^2 = 0$

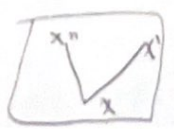
so $\forall_i, x_i = x_i' \Rightarrow x = x'$

Proof (of (2)): $d(x, x') = \sqrt{\sum (x_i' - x_i)^2} = \sqrt{\sum (x_i - x_i')^2} = d(x', x)$

Proof (of (3)): $d(x, x'')^2 = \sum (x_i'' - x)^2$

$$(d(x, x') + d(x', x''))^2 = \sum (x_i' - x_i)^2 + \sum (x_i'' - x_i')^2 + 2\, d(x, x') \cdot d(x', x'').$$

Easier proof ..... If you grant me (3) for $n = 2$: note that 3 points x, x', x'' in $\mathbb{R}^n$ $(n \geq 3)$ determine at least one plane containing x, x', x''



another example of a metric space: $\underline{Ex}$ Let $X = \mathbb{R}^n$, and $d_{\ell^\infty} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$(x, x') \longmapsto \max_{i=1,\dots,n} |x_i' - x_i| .$$

This is called the ℓ^∞ metric on $\mathbb{R}^n$

$n = 3$

$x = (7, 12, 1)$ $\qquad d_{\ell^\infty}(x, x') = \max\{|1-7|, |0-12|, |4-1|\}$

$x' = (1, 0, 4)$ $\qquad = \max\{6, 12, 3\} = 12$

Definition: Let (X, d) be a metric space. Fix $x \in X$ and $r \in \mathbb{R}$ w/ $r > 0$. The open ball of radius r centered at x is the set

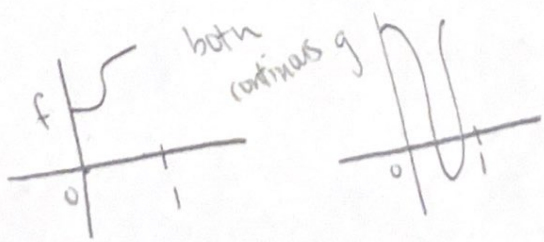
$$\mathrm{Ball}(x, r) = \{x' \in X \mid d(x, x') < r\}$$

definition: Let (X, d) be a metric space. the metric topology on X (or, the topology induced by d) is $\{U \subset X \mid U$ can be written as a union of open balls$\}$

Let $X = \{f : [0,1] \to \mathbb{R} \mid f$ is continuous$\}$

define $d : X \times X \to \mathbb{R}$

$$(f, g) \longmapsto \int_0^1 |g(x) - f(x)| \, dx$$



Proposition: d is a metric on X (L^1 metric on X).

Proof of (1): If $f=g$, $d(f,g) := \int_0^1 |g(x)-f(x)|\,dx = \int_0^1 0\,dx = 0$

If $f \neq g$, then $d(f,g)$, $f \neq g \Rightarrow \exists\, x \in [0,1]$ s.t. $|g(x)-f(x)| > 0$