Questions: 10/24/23

3.) If X is Hausdorff, then the diagonal of X is closed.

Not Haus $\Rightarrow$ Δ NOT closed

Δ closed $\Rightarrow$ Haus

3'.) If the diagonal is closed, then X is Hausdorff.

6.) Not Haus $\Rightarrow$ Δ NOT closed and Δ Not open

6'.) $\Leftarrow$

The contrapositive of "$p \Rightarrow q$" is "$\sim q \Rightarrow \sim p$" ("Not $q \Rightarrow$ Not p")

Contrapositive of 6.) If Δ is closed or open then X is Hausdorff.

Today: Metric Spaces

Informally: A metric space is a set X equipped with a way to measure "the" distance between any two points of X.

(To be able) to give a distance to any pair of points is to give a function $d: X \times X \longrightarrow \mathbb{R}$

$(x, x') \longmapsto d(x, x')$

satisfying:

(Non-degeneracy)

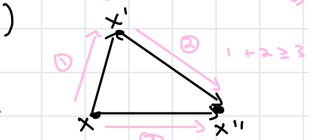
1.) $\forall x \in X$, $d(x, x) = 0$ AND $\forall x, x' \in X$, $d(x, x') = 0 \Rightarrow x = x'$.

(symmetry)

2.) $\forall_{x, x' \in X}$, $d(x, x') = d(x', x)$

(Triangle Inequality)

3.) $\forall x, x', x''$, $d(x, x') + d(x', x'') \geq d(x, x'')$



Definition: A metric space is a set X equipped with a metric d. We often say that (X, d) is a metric space or that X is a metric space (leaving d implicit).

Ex: Let $X = \mathbb{R}^n$, and $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(x, x') \longmapsto \sqrt{\sum_{i=1}^{n} (x_i' - x_i)^2}$$

This is called the standard metric.

Proof 1.): Clearly $d(x,x) = 0$. On the other hand, if $d(x,x') = 0$ then $\sum_{i=1}^{n} (x_i' - x_i)^2 = 0$. So $\forall i, x_i = x_i' \Rightarrow x = x'$.

Proof 2.): $d(x,x') = \sqrt{\sum (x_i' - x_i)^2} = \sqrt{\sum (x_i - x_i')^2} = d(x',x)$

Proof 3.): 1.) $d(x,x'')^2 = \sum (x_i'' - x_i)^2$

2.) $(d(x,x') + d(x',x''))^2 = \sum (x_i' - x_i)^2 + \sum (x_i'' - x_i')^2 + 2\, d(x,x')\, d(x',x'')$

TBD

If you grant me 3.) for $n=2$: Note that three points x, x', x'' in $\mathbb{R}^n$ $(n \geq 3)$ determine at least one plane containing x, x', x''.

(Only for standard metric)

Ex: Let $X = \mathbb{R}^n$, and $d_{\infty} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(x, x') \longmapsto \max_{i=1,\dots,n} |x_i' - x_i|$$

This is called the l^{∞} metric on $\mathbb{R}^n$.

$n=3$: $x = (7, 12, 1)$ $x' = (1, 0, 4)$

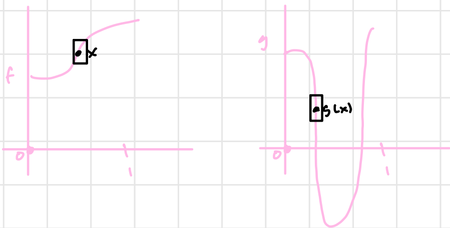
$d_{\infty}(x,x') = \max\{|1-7|, |0-12|, |4-1|\} = \max\{6, 12, 3\} = 12$

Definition: Let (X,d) be a metric space. Fix $x \in X$ and $r \in \mathbb{R}$ with $r > 0$. The open ball of radius r centered at x is the set satisfying

$$\mathrm{Ball}(x,r) = \{x' \in X \mid d(x,x') < r\}.$$

Definition: Let (X,d) be a metric space. The metric topology on X (or, the topology induced by d) is $\{U \subset X \mid U$ can be written as a union of open balls$\}$

Ex: Let $X = \{f : [0,1] \to \mathbb{R} \mid f$ is continuous$\}$



Define $d : X \times X \to \mathbb{R}$

$$(f,g) \longmapsto \int_0^1 |g(x) - f(x)|\,dx$$

Proposition: d is a metric on X

(L^1 metric on X)

Proof 1.): If $f = g$, $d(f,g) = \int_0^1 |g(x) - f(x)|\,dx$

$$= \int_0^1 0\,dx = 0$$

If $f \neq g$, then $d(f,g) > 0$. $f \neq g \Rightarrow \exists_{x \in [0,1]}$ such that $|g(x) - f(x)| > 0$.