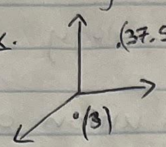
Recall: given a function $f: \mathbb{R}^m \to \mathbb{R}^n$

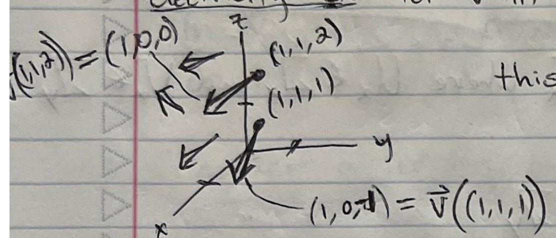
$x \longmapsto f(x)$

ex. $m=3$ $n=1$



f assigns a single value to points in $\mathbb{R}^3$ (i.e temp. at that spot in a 3D room)

electricity ex let $\vec{V}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.



this is an electric field.

Electric potential

$U: \mathbb{R}^3 \to \mathbb{R}$ gradiant of U $\to$ $\nabla U = \vec{V}$ $\leftarrow$ the electric field

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$f: \mathbb{R}^m \to \mathbb{R}^n$

$x \longmapsto f(x) = (f_1(x), f_2(x), f_3(x) \ldots, f_n(x))$

ex. $m=1, n=3$ $f(t) = (t, \cos t, \sin t) = (f_1(t), f_2(t), f_3(t))$

$\boxed{\text{f is cts. iff each } f_1, f_2, \ldots, f_n \text{ are cts.}}$ for $f: \mathbb{R}^m \to \mathbb{R}^n$

Note $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ copies}}$

We can generalize as follows:

A function $f: W \to X \times Y$ is continuous iff each component

$p: W \to X$, $q: W \to Y$ are continuous.

Def. Product topology

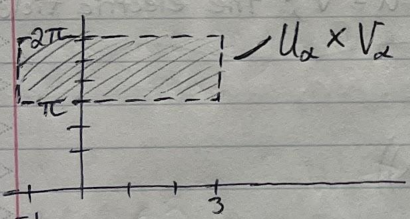
Let X, Y be topological spaces. The product topology

$$\mathcal{T}_{X\times Y} := \{ U \subseteq X\times Y \mid \underbrace{U \text{ can be written as a union of products of open subsets of } X \text{ and } Y \text{ respectively.}}_{U = \bigcup_{\alpha\in A} U_\alpha \times V_\alpha \text{ where } U_\alpha \in \mathcal{T}_X \text{ and } V_\alpha \in \mathcal{T}_Y}$$

or "product"

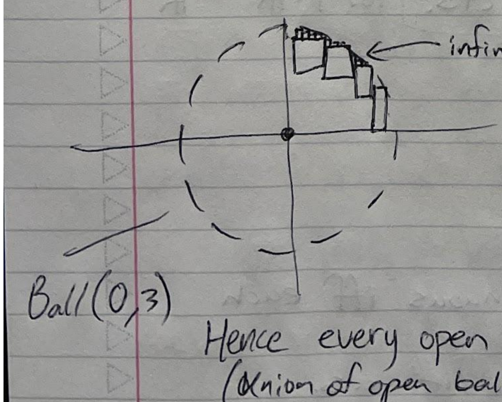
ex. $X = Y = \mathbb{R}$ $\quad U_\alpha \in \mathcal{T}_X$ and $V_\alpha \in \mathcal{T}_Y$

$U_\alpha = (-1, 3) \subseteq \mathbb{R}$ $\qquad V_\alpha = (\pi, 2\pi) \subseteq \mathbb{R}$



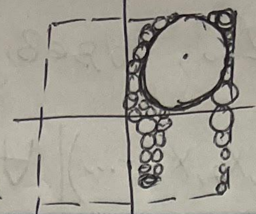
notice $U_\alpha \times V_\alpha$ is open in $\mathbb{R}^2$

So any $U \subseteq X \times Y$ is open if it can be written as a union of open rectangles. (this could even be circular!)



Hence every open element in the standard topology (union of open balls) is also open in the product topology.

$$\mathcal{T}_{std} \subseteq \mathcal{T}_{product}$$



furthermore, for any point in the open square, there is an open ball contained in the square ~~[illegible]~~ containing such point. So $\mathcal{T}_{product} \subseteq \mathcal{T}_{std}$

Hence $\mathcal{T}_{std} = \mathcal{T}_{product}$. By induction we can show this is true for $\mathcal{T}_{std} = \mathcal{T}_{product}$ on any $\mathbb{R}^n$

(in $\mathbb{R}^1$ there is no distinction between open balls and open rectangles)

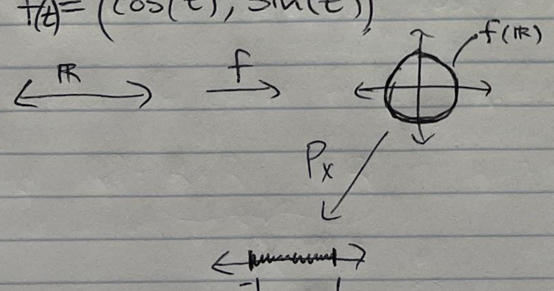
<u>ex.</u> Consider a topology on $[5] \times \mathbb{R}^3$

Projection map $p_X : X \times Y \to X$, $(x,y) \mapsto x$ and $p_Y : X \times Y \to Y$, $(x,y) \mapsto y$

<u>THM</u> (part of) the Universal property of product topology
a function $f : W \to X \times Y$ is cts. iff

$p_X \circ f$ and $p_Y \circ f$ are continuous.

<u>ex</u> $f(t) = (\cos(t), \sin(t))$



<u>Def</u> Fix a collection of spaces $\{X_\beta\}_{\beta \in B}$. The product topology on $\underbrace{\prod_{\beta \in B} X_\beta}_{\text{infinite direct product}}$ (an element of $\prod_{\beta \in B} X_\beta$ is a sequence $(x_\beta)_{\beta \in B}$

ex. Let $B = \mathbb{Z}_{>0}$, $\forall \beta \in B$, let $X_\beta = \mathbb{R}$, then

$$\prod_{\beta \in \mathbb{Z}_{>0}} X_\beta = \{(x_1, x_2, x_3, \dots) \mid \forall \beta \in \mathbb{Z}_{>0}, x_\beta \in \mathbb{R}\}$$

The topology on $\prod_{\beta \in B} X_\beta$ is the collection of subsets that can be written as a union of sets of the form

$$\prod_{\beta \in B'} X_\beta \times \prod_{\beta \in B''} U_\beta \quad \text{where } B'' \text{ is finite, } B' \cup B'' = B.$$

$\mathbb{R} \times \mathbb{R} \times (-3,1) \times (1,5) \times \mathbb{R} \times \mathbb{R} \times \dots$ is open in our ex.

$(-1,1) \times (-1,1) \times (-1,1) \times (-1,1) \times \dots$ is <u>not</u> open in our ex.

there are only finitely many "U_β" ($(-3,1), (1,5)$ in example 1)
So it's open.

A function $f: W \to \prod_{\beta \in B} X_\beta$ is cts. iff $P_\beta \circ f$ is cts for all $\beta \in B$.