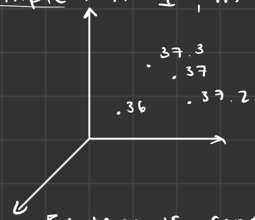
Date: October 12, 2023

Example: $n = 1$, $m = 3$

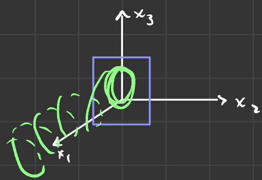


Ex: temp is continuous.

Recall: (from cal 3) Given a function

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n, x \longmapsto f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$.
f is continuous if and only if each $f_1, f_2, \ldots, f_n$ is continuous.

Example: $m = 1$, $n = 3$, $f(t) = (t, \cos(t), \sin(t))$ // Helix // Then $f_1(t) = t$, $f_2(t) = \cos(t)$, $f_3(t) = \sin(t)$



Note: $\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n\text{-copies}}$.

Today: Theres a generalization of ★.
Proposition: A function $f : W \longrightarrow X \times Y$ is continuous iff each component of f is continuous.

Definition (product topology): Let X and Y be topological spaces. Then the product topology on $X \times Y$ is

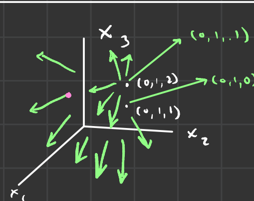
$$\tau_{X \times Y} := \{ U \subset X \times Y : U \text{ can be written as a union} \ldots$$

$$\tau_{product} \quad \ldots U = \bigcup_{\alpha \in A} U_\alpha \times V_\alpha \text{ where each } U_\alpha \text{ is in } \tau_X, \text{ and each} \ldots$$

$$\ldots V_\alpha \in \tau_Y \}$$

Another example: $\vec{v} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

An electric field is such a function.



$U = \bigcup_{\alpha \in A} \emptyset \times \emptyset$

- empty set

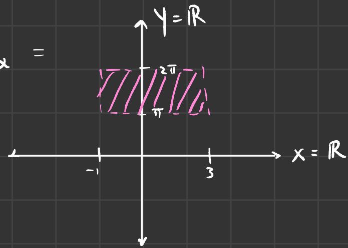
$U : \mathbb{R}^3 \longrightarrow \mathbb{R}$

↳ electrical potential (voltage) Turns out: $\nabla U = \vec{v}$

Example: $X = Y = \mathbb{R}$

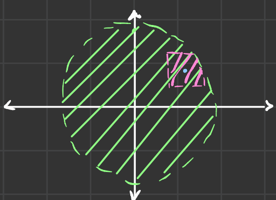
- An example of a $U_\alpha \in \tau_X$ is $(-1,3) \subset \mathbb{R}$
- " " " " $V_\alpha \in \tau_Y$ is $(\pi, 2\pi) \subset \mathbb{R}$

Then: $U_\alpha \times V_\alpha =$

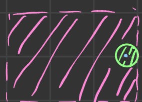


Because $U_\alpha \times V_\alpha = \{ (x_1, x_2) : x_1 \in (-1,3) \ \& \ x_2 \in (\pi, 2\pi) \}$

Ex: $\mathrm{Ball}(\overset{\text{origin}}{0}, 3) \subset \mathbb{R}^2$ is open the product topology.



Hence $\tau_{std} \subset \tau_{product}$.

In fact, open rectangles  are open in τ_{std} so $\tau_{std} \supset \tau_{product}$.

Thm. τ_{std} on $\mathbb{R}^n$ is $\tau_{product}$ on $\mathbb{R}^n$. $\mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n}$

$\cong \mathbb{R} \times (\mathbb{R}^{n-1})$

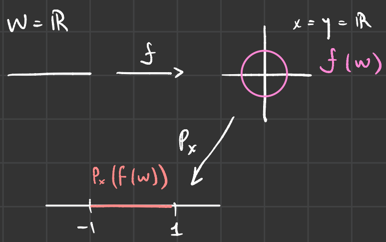
Ex: We now have an (interesting) topology on $[5] \times \mathbb{R}^3$.

Thm (part of universal property of product topology): A function

$W \xrightarrow{f} X \times Y$ is continuous iff $p_x \circ f$ and $p_y \circ f$ are continuous.

Notation: $(x,y) \mapsto x$ $\quad X \times Y \xrightarrow{p_x} X$, $X \times Y \xrightarrow{p_y} Y$ $\quad (x,y) \mapsto y$

Example: $t \longmapsto (\cos(t), \sin(t))$



Defn: Fix a collection of spaces $\{X_\beta\}_{\beta \in B}$. The product topology on $\prod_{\beta \in B} X_\beta$ ... is the collection of subsets that can be written as a union of sets of the form

→ An (infinite) direct product of sets: An element of $\prod_{\beta \in B} X_\beta$ is a choice of $x_\beta \in X_\beta$ for every $\beta \in B$.

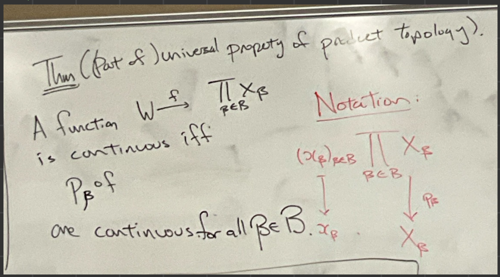
$$\prod_{\beta \in B'} X_\beta \times \prod_{\beta \in B''} U_\beta$$ where B'' is finite, $B' \cup B'' = B$.

EX: $B = \mathbb{Z}$. $\forall \beta$ let $X_\beta = [-|\beta|, |\beta|]$. An element of $\prod_{\beta \in \mathbb{Z}} X_\beta$ is a sequence $(x_\beta)_{\beta \in \mathbb{Z}}$ such that $\forall \beta \in \mathbb{Z}$, $x_\beta \in [-|\beta|, |\beta|]$.

EX: Let $B = \mathbb{Z}_{\geq 1}$, $\forall \beta \in B$, let $X_\beta = \mathbb{R}$. Then $\prod_{\beta \in \mathbb{Z}_{\geq 1}} X_\beta = \{(x_1, x_2, x_3, x_4, \ldots) : \forall \beta \in \mathbb{Z}_{\geq 1}, x_\beta \in \mathbb{R}\}$

$\mathbb{R} \times \mathbb{R} \times (-3,1) \times (1,5) \times \mathbb{R} \times \mathbb{R} \times$ is open

$(-1,1) \times (-1,1) \times (-1,1) \times (-1,1) \times \ldots$ is NOT open.


Thm (Part of) universal property of product topology).

A function $W \xrightarrow{f} \prod_{\beta \in B} X_\beta$ is continuous iff $p_\beta \circ f$ are continuous for all $\beta \in B$.

Notation:

$$\begin{array}{ccc} (x_\beta)_{\beta \in B} & & \prod_{\beta \in B} X_\beta \\ \downarrow & & \downarrow p_\beta \\ x_\beta & & X_\beta \end{array}$$