Compactness II

defn: a topological space X is called compact if every cover of X admits a finite subcover

(not if X admits a finite subcover, every cover needs to admit one)

thm: (Heine-Borel Theorem)
a subset $\underbrace{A \subset \mathbb{R}^n \text{ is compact}}$ if and only if $\underbrace{A \text{ is closed}}$ & bounded

as $\subseteq \mathbb{R}^n$

$\mathbb{R}^n$ → standard topology

A needs to be a topological space (subspace topology)

if A^c is open
$A^c = \{x \in \mathbb{R}^n \mid x \notin A\}$
so is $A^c \subset \mathbb{R}$ open?

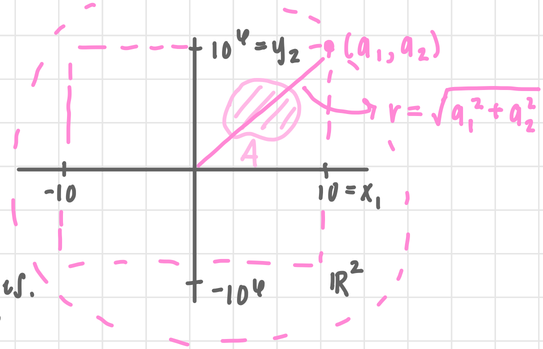
defn: a subset A of $\mathbb{R}^n$ is bounded if $\exists\, r > 0 \in \mathbb{R}$ s.t. $A \subset \text{Ball}(0, r)$

to produce compact spaces, we need closed and bounded subsets of $\mathbb{R}^n$

rmk: shouldn't be too hard to show a set is bounded (or not).

ex. suppose $\exists$ #s $\overset{10}{a_1}, \overset{10^4}{a_2}, \dots, a_n \in \mathbb{R}_{>0}$ s.t. $\forall x \in A \subset \mathbb{R}^n$ $(|x_i| \leq a_i)$
then $\underbrace{A \subset \text{Ball}(0, \textstyle\sum a_i^2)}_{\text{conditions for bounded}}$

ex.



proposition: let $\overset{\mathbb{R}^n}{X}, \overset{\mathbb{R}}{Y}$ be topological spaces.
and $f: X \to Y$ is a continuous function.
if $\underbrace{K \subset Y}_{K \subset \mathbb{R}^n}$ is closed, so is $f^{-1}(K)$ of X

of $\mathbb{R}^n$

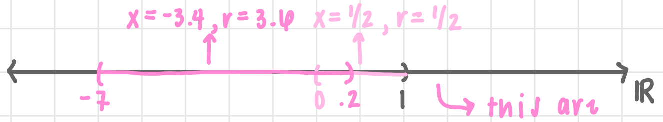
"as a subset of"

ex. $X = \mathbb{R}^3$, $Y = \mathbb{R}$
$f: \mathbb{R}^3 \to \mathbb{R}$ (f is continuous)

$(x_1, x_2, x_3) \longmapsto \underbrace{\sqrt{x_1^2 + x_2^2 + x_3^2}}_{\text{distance from origin}}$

let $K = \{1\} \subset \mathbb{R}$

is K closed?
yes b/c K^c is open
$K^c = (-\infty, 1) \cup (1, \infty)$ can be written as a union of open balls in $\mathbb{R}$



$\text{Ball}(x, r) = (x - r, x + r) \subset \mathbb{R}$

single points in $\mathbb{R}$ are closed

so $\underbrace{f^{-1}(K)}_{\substack{\text{distance of } (x_1, x_2, x_3) \\ \text{from origin} = 1 \\ \text{aka } S^2}}$ is closed

is a subset of an open ball of radius r = 2.

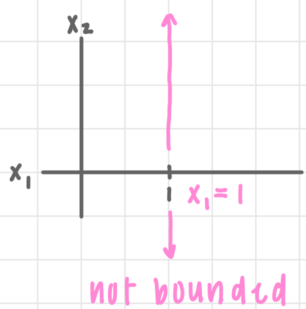
is $f^{-1}(K)$ bounded?
yes b/c a sphere w/ $r=1$ can be bounded by a sphere of $r=2$

... so the Heine-Borel Theorem tells us S^2 is compact

<u>warning</u>: K bounded $\not\Rightarrow$ $f^{-1}(K)$ is bounded

bounded only describes Euclidean spaces

ex. $X = \mathbb{R}^2$, $Y = \mathbb{R}$
let $f: X \to Y$
$(x_1, x_2) \longmapsto x_1$
and $K = \{1\} \subset \mathbb{R}$
then $f^{-1}(K) = \{(1, x_2) \mid x_2 \in \mathbb{R}\}$



to show something is open / closed, try to create a function to $\mathbb{R}$

then use $K \subset Y$ is closed $\Longrightarrow f^{-1}(K)$ is closed