DATE: September 21, 2023 (birthday ahh).

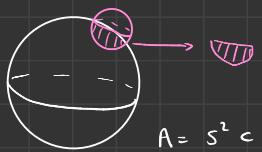
Last time: compactness

Defn. A topological space X is called compact if every open cover of X admits a finite sub-cover.

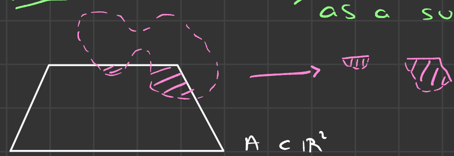
Today: Theorem (Heine-Borel Thm). - A subset $A \subset \mathbb{R}^n$ (std. top.) is compact if and only if A is closed and bounded. (w/ a given subspace top).

as a subset of $\mathbb{R}^n$

with A given the subspace topology.



$A = S^2 \subset \mathbb{R}^3$



$A \subset \mathbb{R}^2$

$A \subset \mathbb{R}^n$ $\quad A^c = \{x \in \mathbb{R}^n : x \notin A\}$, is $A^c \subset \mathbb{R}^n$ open? (a union of balls).

Defn. A subset A of $\mathbb{R}^n$ is bounded if $\exists$ real # $r > 0$ such that $A \subset \text{Ball}(0, r)$. //check if A is bounded & close //.

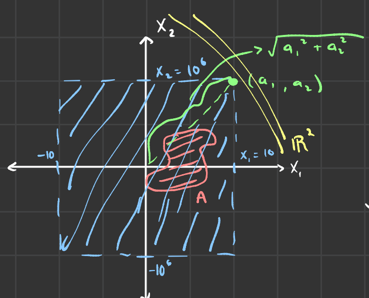
To produce examples of compact spaces, we need only produce examples of closed + bounded subsets of $\mathbb{R}^n$.

Rmk. In many situations, it wont be too hard to show a set A is bounded (or not).

Ex. Suppose $\exists$ #'s $a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}$ such that $\forall x \in A \subset \mathbb{R}^n$.

$|x_i| \leq a_i$ Then $A \subset \text{Ball}(0, \sum a_i^2)$

Ex.



topological spaces

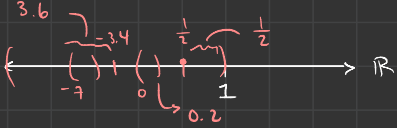
Proposition: Let X, Y be topology spaces, and ($\mathbb{R}^n$ $\mathbb{R}$)

$f: X \longrightarrow Y$ ($\mathbb{R}^n \to \mathbb{R}$) a continuous function. If $K \subset Y$ ($K \subset \mathbb{R}$) is closed, so is $f^{-1}(K)$ of X (of $\mathbb{R}^n$)

Ex. $X = \mathbb{R}^3$, $Y = \mathbb{R}$ $\quad f : \mathbb{R}^3 \to \mathbb{R} \quad (x_1, x_2, x_3) \mapsto \sqrt{x_1^2 + x_2^2 + x_3^2}$

// whats the distance from the origin, and is continuous //

Let $K = \{1\} \subset \mathbb{R}$ // $(-\infty, 1) \cup (1, \infty) = K^c$ //

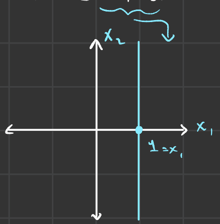


Ball $(x, r) = (x - r, x + r) \subset \mathbb{R}$

K is closed, so $f^{-1}(K)$ is closed. So S^2 is closed. The sphere is compact because is closed and bounded.

Rmk | warning: K bounded $\not\Rightarrow$ $f^{-1}(K)$ bounded.

Ex. $X = \mathbb{R}^2$, $Y = \mathbb{R}$. Let $f : (x_1, x_2) \mapsto x_1$ and $K = \{1\} \subset \mathbb{R}$. Then
$f^{-1}(K) = \{(1, x_2) : x_2 \in \mathbb{R}\}$ // A line from $x_1 = 1$ //



¿Quiz? - Now we have a quiz every day.