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Last Time: Def A "topology" $\mathcal{T}$ on X is a collection of subsets of X satisfying:

i) $\emptyset, X \in \mathcal{T}$

$\mathcal{T}$ is closed under finite $\cap$ → ii) for any finite collection $U_1, \dots, U_k \in \mathcal{T}$, $\bigcap_{i=1,\dots,k} U_i \in \mathcal{T}$

$\mathcal{T}$ is closed under $\cup$ → iii) for any collection $U_1, \dots, U_k \in \mathcal{T}$, $\bigcup_{i=1} U_i \in \mathcal{T}$

ex. Let $X = \{0,1,2,3\}$.

Let $\mathcal{T} = \left\{ \begin{matrix} \{2\} & \{0,1,2\} \\ \{1,2\} & \emptyset \end{matrix} \right\}$

Claim: $\mathcal{T}$ satisfies (ii) and (iii)

(ii) $\{2\} \cap \{\emptyset\} = \emptyset \in \mathcal{T}$ $\quad \{0,1,2\} \cap \{1,2\} \cap \{2\} = \{2\} \in \mathcal{T}$

Check the rest of intersections between elements of $\mathcal{T}$.

$\mathcal{T}$ satisfies (ii) and (iii)

Since $X = \{0,1,2,3\} \notin \mathcal{T}$, $\mathcal{T}$ doesn't satisfy (i) so isn't a topology.

ex. Let $X = \mathbb{R}^2$ $\quad \mathcal{T} := \{ U \subseteq X \mid U \text{ is open in } X \}$

Last time: $\mathcal{T}$ satisfies (i), (ii), (iii).

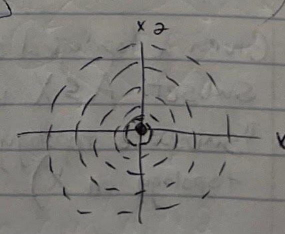
Lets see why (ii) fails when finite is omitted.

Consider $\{Ball(0,r)\}_{r \in \mathbb{R}, r>0}$ $\quad$ (i.e. $A = \mathbb{R}_{>0}$ $\quad U_\alpha := Ball(0,\alpha)$)

origin

Then $\bigcap_{\alpha \in A} U_\alpha = \bigcap_{r \in \mathbb{R}_{>0}} Ball(0,r) = \{0\}$

notice that this intersection is not open.



intersection is the origin.

So why does (iii) hold?

fix $U_1, \dots, U_k \in \mathcal{T}$ and fix $x \in \bigcap_{i=1,\dots,k} U_i$

WTS: $\exists r \in \mathbb{R}_{>0}$ s.t. $Ball(x,r) \subseteq \bigcap_{i=1}^{K} U_i$

proof: Each U_i is open, so $\exists r_i$ s.t. $Ball(x, r_i) \subseteq U_i$
Let $r = \min\{r_1, \dots, r_k\}$ then $Ball(x,r) \subseteq \bigcap_{i=1,\dots,K} U_i$ ✓

This is why finitely many U's is read.

Def: $f: X \to Y$ is cts. iff the preimage of an open subset is open

Def: $K \subseteq X$ is closed iff $K^c \in \mathcal{T}$
open iff $K \in \mathcal{T}$

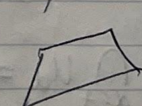
Def: a topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X.

Def: a cts. function $f: X \to Y$ is a homeomorphism if

(1) f is a bijection (able to go back and forth)
(2) f^{-1} is cts. (back and forth must be cts.)

Subspace Topology

Most shapes are naturally subsets of $\mathbb{R}^n$ (for some n)

 $\subseteq \mathbb{R}^2$

Given a topological space X, we'll endow any subset $A \subseteq X$ with a topology. called the "subspace topology" on A (which depends on/inherits something from the topology on X)

Def: Fix a topological space $(X, \mathcal{T})$ and a subset $A \subseteq X$.

The subspace topology on A is

$$\mathcal{J} = \{ V \subseteq A : \exists U \in \mathcal{T} \text{ s.t. } V = U \cap A \}$$

J doesn't need to be unique

This should say that U need not be unique. The subspace topology (fancy T) *is* unique.

(i) $\emptyset \in \mathcal{J}$ by picking $U = \emptyset$, $A \in \mathcal{J}$ by letting $U = X$.

Recall: Fix $A \subseteq X$. Then there's a natural inclusion function:

$$\begin{aligned} i_A : A &\to X \\ x &\mapsto x \end{aligned}$$

THM (Universal Property of the subspace topology)
Let X be a topological space and fix a subset $A \subseteq X$.
The subspace topology satisfies:

(1) $i_A : A \to X$ is cts.

(2) suppose $f : W \to X$ is cts. and $f(W) \subseteq A$.
then $\begin{aligned} f' : W &\to A \\ w &\mapsto f(w) \end{aligned}$ is cts. and $\underbrace{i_A \circ f' = f}$

(3) f' is the only function satisfying this eqn.