9/14/2023 Notes on Topology - Wk 4-2

- Recall that a topology $\mathcal{T}$ on X is a collection of subsets of X satisfying
  - $\phi, X \in \mathcal{T} \longrightarrow (1)$
  - $\mathcal{T}$ is closed under finite intersections $\longrightarrow (3)$
  - $\mathcal{T}$ is closed under unions $\longrightarrow (2)$

Ex.) Let $X = \{0,1,2,3\}$

Let $\mathcal{T} = \{\{2\}, \{0,1,2\}, \{1,2\}, \phi\}$.

Then $\mathcal{T}$ satisfies (2) and (3)

$\{2\} \cap \phi = \phi \in \mathcal{T}$

$\{0,1,2\} \cap \{1,2\} \cap \{2\} = \{2\} \in \mathcal{T}$

Ex.) Let $X = \mathbb{R}^2$. Let $\tau := \{ U \in X \mid U$ is open in $X \}$

We know that τ satisfies (1), (2), (3).

Yet (3) would fail if "finite" is omitted. Why?

→ Consider the collection $\{ \text{Ball}(0, r) \}_{r \in \mathbb{R}_{>0}}$

(i.e. $A = \mathbb{R}_{>0}$, $U_a := \text{Ball}(0, a)$)

Then $\bigcap_{\alpha \in A} U_\alpha = \bigcap_{r \in \mathbb{R}_{>0}} \text{Ball}(0, r) = \{0\}$

Why does (3) hold?

Let $U_1, \dots, U_k \in \tau$. Also let

$x \in \bigcap_{r=1,\dots,k} U_i$.

We want to show that $\exists\, r \in \mathbb{R}, r > 0$

s.t. $\text{Ball}(x, r) \subset \bigcap_{i=1}^{k} U_i$. ↓

Let $r = \min\{r_1, \dots, r_k\}$.

Then $\forall i$, $Ball(x,r) \subset Ball(x, r_i)$

$\subset U_i$.

Hence $Ball(x,r) \subset \bigcap_{i=1,\dots,k} U_i$

. Recall that $f: X \to Y$ is continuous iff the pre-image of an open subset is open.

(Recall) . $K \subset X$ is closed iff $K^c \in$ __

and open iff $K \in$ __

(Recall) . A topological space is a pair where $\mathcal{T}$ is a topology on

(Recall) . A continuous function $f: X \to Y$ is a homeomorphism if

$\downarrow$

- Discrete Topology
- Trivial Topology

- f is a bijection, and
- f^{-1} is continuous

Subspaces

- Most shapes are naturally subsets of $\mathbb{R}^n$ (for some n).

Ex.)  $\subset \mathbb{R}^2$, where ▱ = trapezoid

Given a top space X, we'll endow any subset $A \subset X$ with a topology. We'll also characterize this topology by a "universal property," which is called "the subspace topology" on A.

- Note that the subspace topology depends on the topology of X

- Fix a topological space $(X, \mathcal{T})$, a subset $A \subset X$. The subspace topolo

$\{V \subset A \mid \exists U \in \mathcal{T}$ s.t. $V =$

- Note that U need not be unique

- Recollection: Fix $A \subset X$.

  Then there exists a natural <u>inclusion function</u>

"iota" $\longrightarrow$ $i_A : A \longrightarrow X$

$x \longmapsto x$

- Theorem (Universal Property of the Subspace Topology):

  Let X be a topological space, and fix a subset A of X.

  The subspace topology satisfies

  1) $i_A : A \longrightarrow X$ is continuous

  2) Suppose $f : W \longrightarrow X$ is continuous, and $f(W) \subset A$.

     Then the function $f' : W \rightarrow A$

     $w \longmapsto f(w)$

$\downarrow$

is continuous, and $i_A \circ f' = f$

3) f' is the only function satisfying