Questions of the Day:

1. Is "U_α" always an open ball?

   No

2. What is the exam format/structure?

   mostly multiple choice, T/F, definitions, & 1-2 proofs

3. If U is open (in $\mathbb{R}^n$) is $U \setminus \{\infty \text{ many points}\}$ still open? how about $U \setminus$ some infinite set?

   Yes. Depends.

4. Compare "closed" and "$\underbrace{\text{having a border}}_{\text{not defined for us yet}}$"

   (usually, they mean the same)

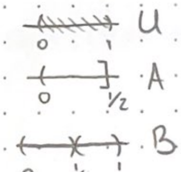
Review: When we are given $A \to P(X)$ we often write $\alpha \in A \mapsto U_\alpha \subset X$

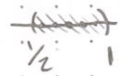
ex for Q3:

$n=1$, $\mathbb{R}^n = \mathbb{R}$, $(0,1) \subset \mathbb{R}$

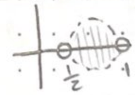
let $A = (0, \frac{1}{2}]$

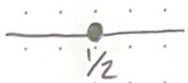
$B = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$

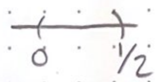


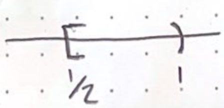
$U \setminus A$  open bc every point has an open ball

$$\boxed{\text{remark: consider } \{(t,0) \mid t \in (\tfrac{1}{2}, 1) \subset \mathbb{R}\} \subset \mathbb{R}^2}$$



$U \setminus B$  not open bc you can't make a ball around it

let $C =$ 

$U \setminus C =$  neither open nor closed

6. Is $\mathbb{R}^0$ open?

   as a subset of $\mathbb{R}^0$, yes.

So far: Defined a notion of "open subset" for:

- $\mathbb{R}^n$
- posets

What unifies these examples?

$\mathbb{R}^n$ and a poset P are both topological spaces

Def. of Topology: fix a set X. fix a collection $\mathcal{T}$ of subsets of X. $\mathcal{T}$ is called a topology on X iff the following hold:

1. $\emptyset, X \in \mathcal{T}$
2. for any collection $\{U_\alpha\}_{\alpha \in \mathcal{J}}$ of elements in $\mathcal{T}$ $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$
3. for any finite collection $U_1, \ldots, U_k \in \mathcal{T}$ the intersection $\bigcap_{i=1\ldots k} U_i$ is an element of $\mathcal{T}$

ex) $X = \mathbb{R}^n$

$\mathcal{T} := \{U \subset X \mid U \text{ is open}\}$

ex) X is a poset

$\mathcal{T} := \{U \subset X \mid U \text{ is open}\}$

thm (hw 2): let X be a poset. Define $\mathcal{T} := \{U \subset X \mid \forall p \in U, q \in X \text{ st } p \leq q, q \in U\} = \{\text{If } p \in U, q \in X \text{ and } p \leq q, \text{ then } q \in U\}$ then $\mathcal{T}$ is a topology on X (this is called the Alexandroff topology).

thm: fix $n \geq 0$. let $X = \mathbb{R}^n$. Define $\mathcal{T} := \{U \subset X \mid U \text{ is a union of open balls}\}$. Then $\mathcal{T}$ is a topology on X (this $\mathcal{T}$ is called the Standard topology on $\mathbb{R}^n$)

Def: a topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X. Given a topological space we say $U \subset X$ is open iff $U \in \mathcal{T}$. A subset $K \subset X$ is called closed if $K^c \in \mathcal{T}$.

Def: let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f: X \to Y$ is called continuous if $\forall\ V \in \mathcal{T}_X$, $f^{-1}(V) \in \mathcal{T}_X$.

Recall: $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$

Def: A continuous function $f: X \Rightarrow Y$ is called a homeomorphism if

1. f is a bijection and
2. f's inverse function is continuous