Math 4330
Topology
Sept 5^{th}

<u>Topic of Dicussion:</u> Open subsets in $\mathbb{R}^n$

Fix an integer $n \geq 0$, and a subset $U \subset \mathbb{R}^n$.

Ex: $n=1$ 

$n=2$ $\emptyset \subset \mathbb{R}^2$

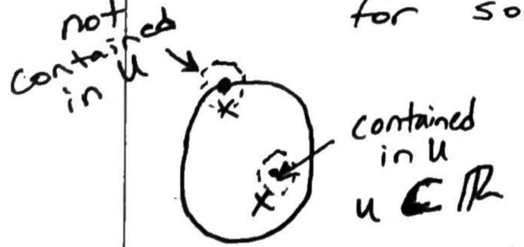
The following are equivalent:

(a) U can be written as a union of open balls.

(b) $\forall x \in U$, $\exists$ $r > 0$ ($r \in \mathbb{R}$) s.t. $Ball(x, r) \subset U$.

<u>Recall:</u> An open ball in $\mathbb{R}^n$ is a set of the form:

$$Ball(x, r) := \{ y \in \mathbb{R}^n \text{ s.t. } dist(x, y) < r \}$$
for some $x \in \mathbb{R}^n$, $r > 0$.



<u>Recall:</u> For statement (a) to be equal to statement (b) means:

(a) $\Rightarrow$ (b)

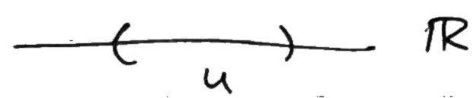
&

(b) $\Rightarrow$ (a)

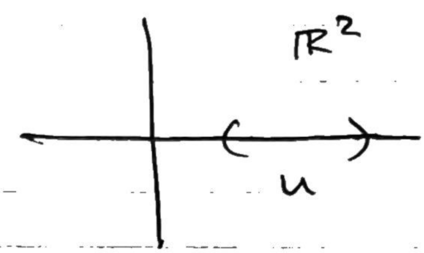
<u>Definition:</u> A subset $\mathcal{U} \subset \mathbb{R}^n$ is called <u>open</u> if $\mathcal{U}$ satisfies either (a) or (b).

⚠ Whether $\mathcal{U}$ is open depends on n (as in $\mathbb{R}^n$).
(Openness is very dimension dependent.)

<u>Visual Example:</u>



open in $\mathbb{R}$



NOT open in $\mathbb{R}^2$

(just a slice or sliver)

<u>Proof of Proposition:</u>

(Need to prove both $a \Rightarrow b$ & $b \Rightarrow a$)

Let's prove (b) $\Rightarrow$ (a) first.

By (b), we know that $\forall x, \exists\, r_x > 0$ s.t. $\text{Ball}(x, r_x) \subset \mathcal{U}$.

<u>Claim:</u> $\underbrace{\bigcup_{x \in \mathcal{U}} \text{Ball}(x, r_x)}_{LHS} = \underbrace{\mathcal{U}}_{RHS}$.

[In order to prove they are equal, we will show they are subsets of each other.]

Proof of Claim: (continuation of (b) $\Rightarrow$ (a))

LHS $\subset$ RHS:

$y \in \bigcup_{x \in U} Ball(x, r_x) \Rightarrow \exists\, x \in U$ st $y \in Ball(x, r_x)$

But by constroction, $Ball(x, r_x) \subset U$.
So $y \in U$.

RHS $\subset$ LHS:

Fix $x \in U$. Then $x \in Ball(x, r_x)$

Now Let's Prove (a) $\Rightarrow$ (b).

By (a), we know that there is some indexing set A, and a $\{(x_\alpha, r_\alpha)\}, \alpha \in A$

st. $U = \bigcup_{\alpha \in A} Ball(x_\alpha, r_\alpha)$

Fix some $x \in U$. We must exhibit some reall # $r > 0$ s.t. $Ball(x, r_x) \subset U$.

By assumption, $\exists$ for some α s.t. $x \in Ball(x_\alpha, r_\alpha)$

Choose r so that $r + dist(x, x_\alpha) < r_\alpha$.
Then $\forall\ y \in Ball(x, r)$

$\text{dist}(y, x_\alpha) \le \text{dist}(y, x) + \text{dist}(x, x_\alpha)$

$< r + \text{dist}(x, x_\alpha)$

(Three components $\nearrow \rightarrow \rightarrow$)

$< r_\alpha$

Rewritten:

$$\text{dist}(y, x_\alpha) \le \text{dist}(y, x) + \text{dist}(x, x_\alpha) < r + \text{dist}(x, x_\alpha) < r_\alpha$$

(Triangle Inequality)

Thus $y \in \text{Ball}(x_\alpha, r_\alpha)$

i.e. $\text{Ball}(x, r) \subset \text{Ball}(x_\alpha, r_\alpha)$.

Since $\text{Ball}(x_\alpha, r_\alpha) \subset U$, we conclude:

$$\text{Ball}(x, r) \subset U.$$

$$U = \bigcup_{\alpha \in \mathcal{A}} \text{Ball}(x_\alpha, r_\alpha)$$

Remark: The proof of this proposition (and its statement) only involve a notation of distance satisfying the triangle inequality. So the proposition must be true for any set (not just $\mathbb{R}^n$) with such structure.