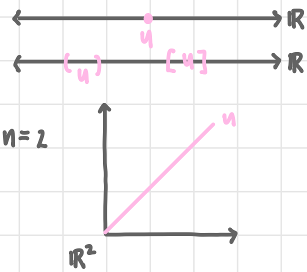
open subsets of $\mathbb{R}^n$

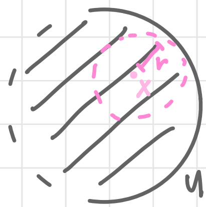
note: different than HW, open subset of posets

ex. $n=1$



prop: fix an integer $n \geq 0$, and a subset $U \subset \mathbb{R}^n$
the following are equivalent:
(a) U can be written as a union of open balls

recall: an open ball in $\mathbb{R}^n$ is a set of the form
$\text{Ball}(x,r) := \{ y \in \mathbb{R}^n \text{ s.t. } \text{dist}(x,y) < r \}$
ex. $\text{Ball}(x,2) =$



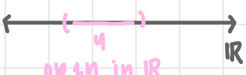
need an r that's contained in u

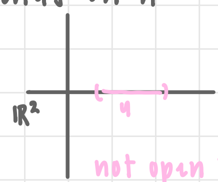
(b) $\forall x \in U, \exists r > 0 \, (r \in \mathbb{R})$ s.t. $\text{Ball}(x,r) \subset U$

for statement (a) to be equivalent to (b) means
(a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a)

defn: a subset $U \subset \mathbb{R}^n$ is called open if U satisfies either (a) or (b).

whether U is open depends on n





not open in $\mathbb{R}^n$ b/c there is not a ball that could fit in U

proof: (of prop) wts that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a)

wts (b) $\Rightarrow$ (a) we assume (b) and show (a)
by (b), we know that $\forall x, \exists r_x > 0$ s.t. $\text{Ball}(x, r_x) \subset U$

claim: $\bigcup_{x \in U} \text{Ball}(x, r_x) = U$

how to show sets are equal:
- they have the same elements
- their subsets of each other (easier way)

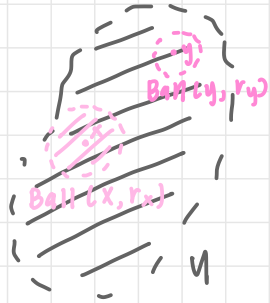
proof of claim: LHS $\subset$ RHS

$y \in \bigcup_{x \in U} \text{Ball}(x, r_x) \Rightarrow \exists x \in U$ s.t. $y \in \text{Ball}(x, r_x)$

but by construction $\text{Ball}(x, r_x) \subset U$

so $y \in U$

start w/ element in LHS, wts it is in RHS



RHS $\subset$ LHS

fix $x \in U$, then $x \in \text{Ball}(x, r_x)$

you can pick any x and make a Ball around it that is in U.

wts (a) $\Rightarrow$ (b)

by (a) we know there is some indexing set A and a collection $\{(x_\alpha, r_\alpha\}_{\alpha \in A}$ s.t. $U = \bigcup_{\alpha \in A} \text{Ball}(x_\alpha, r_\alpha)$

fix some $x \in U$ wts an $r > 0$, $r \in \mathbb{R}$, s.t. $\text{Ball}(x, y) \subset U$

by assumption, $\exists$ some α s.t. $x \in \text{Ball}(x_\alpha, r_\alpha)$

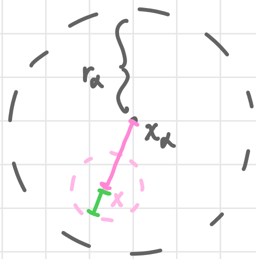
choose an r s.t. $r + \text{dist}(x, x_\alpha) < r_\alpha$

then $\forall y \in \text{Ball}(x, r)$

$$\text{dist}(y, x_\alpha) \leq \text{dist}(y, x) + \text{dist}(x, x_\alpha)$$

(triangle inequality)

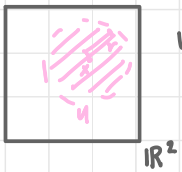
$$< r + \text{dist}(x, x_\alpha)$$
$$< r_\alpha$$



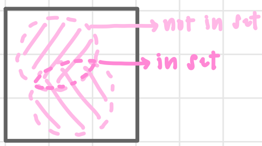
then $y \in \text{Ball}(x_\alpha, r_\alpha)$ i.e. $\text{Ball}(x, r) \subset \text{Ball}(x_\alpha, y_\alpha)$

since $\text{Ball}(x_\alpha, r_\alpha) \subset U$, we conclude $\text{Ball}(x, y) \subset U$ //

remark: the proof of this prop only involve a notation of distance satisfying the $\Delta \leq$, so the prop must be true for any set w/ such structure examples of open subsets:

- $\forall$ integer $n \geq 0$, $\emptyset \subset \mathbb{R}^n$
- $\mathbb{R}^n \subset \mathbb{R}^n$
- 

$U = \text{Ball}(x, r) = \bigcup \text{Ball}(x, r)$

- 

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