Date: September 5, 2023

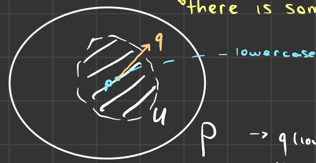
$(P, \leq)$

· $p \neq q$? Defn: Fix a poset P. A subset u of P is called open if:

↳ Whenever p (lower case) is an element of the subset u, and $q \geq p$, we know q is an element of the subset.

Ex: $q = p$

there is some $q \in p$ for which



· $U \in \mathcal{P}(P) \iff U \subset P$

-> q (lowercase) lives inside the poset.

· Upward closed -> you are going up without getting out of the extension.

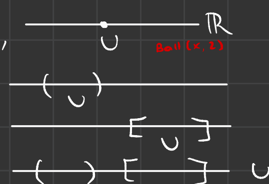
-> u is not opened
-> u needs to be extended.

Open subsets of Euclidean space.

Today: Open subsets of $\mathbb{R}^n$
Hw: Open subsets of a poset.

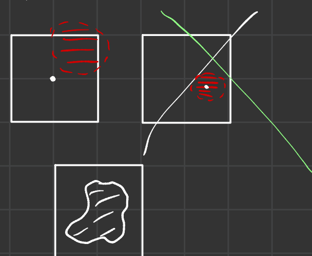
Ex (of U):

$n = 1$,



$n = 2$,

$\emptyset \subset \mathbb{R}^2$



Proposition: Fix $n \geq 0$, and a subset $U \subset \mathbb{R}^n$, The following are equivalent:

a) U can be written as a union of open balls.

Recall: An open ball in $\mathbb{R}^n$ is a set of the form:

$$\text{Ball}(x, r) := \{y \in \mathbb{R}^n : \text{dist}(x, y) < r\} \text{ for}$$

some $x \in \mathbb{R}^n$, $r > 0$

b) $\forall x \in U, \exists r > 0$ $(r \in \mathbb{R})$, such that $\text{Ball}(x, r) \subset U$

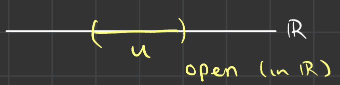
$U \subset \mathbb{R}^2$

Recall: For statement (a) to be equivalent to statement (b) means:

$$(a) \Rightarrow (b) \quad \text{and} \quad (b) \Rightarrow (a)$$

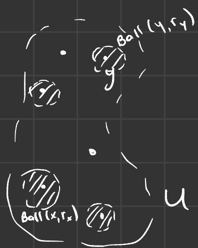
Defn A subset $U \subset \mathbb{R}^n$ is called **open** if U satisfies either (hence both) of (a) & (b).

⚠ Whether U is open depends on n!



proof: (of proposition): Lets prove (b) => (a). By (b), we know that $\forall x$, $\exists\, r_x > 0$ such that $\text{Ball}(x, r_x) \subset U$.

Claim: $\displaystyle\bigcup_{x \in U} \text{Ball}(x, r_x) = U$



Proof of Claim:

- LHS $\subset$ RHS. why?

  $y \in \displaystyle\bigcup_{x \in U} \text{Ball}(x, r_x) \Rightarrow \exists\, x \in U$ such that $y \in \text{Ball}(x, r_x)$

  But by construction, $\text{Ball}(x, r_x) \subset U$.

  so $y \in U$.

- RHS $\subset$ LHS: Fix $x \in U$, then $x \in \text{Ball}(x, r_x)$.

Lets prove (a) => (b). By (a), we know there is some Indexing set A, and a collection $\{(x_\alpha, r_\alpha)\}_{\alpha \in A}$ such that

$$U \in \bigcup_{\alpha \in A} \text{Ball}(x_\alpha, r_\alpha)$$

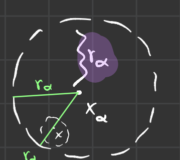
↳ Fix some $x \in U$. We must exhibit some real # $r > 0$ such that $\text{Ball}(x, r) \subset U$. By assumption, $\exists$ some α such that

$$x \in \text{Ball}(x_\alpha, r_\alpha)$$

We choose an r so that:

$$r + \text{dist}(x, x_\alpha) < r_\alpha.$$

Then $\forall\, y \in \text{Ball}(x, r)$.



by choice of r.

Triangular inequality:

$$\text{dist}(y, x) \le \text{dist}(y, x) + \text{dist}(x, x_\alpha) < r + \text{dist}(x, x_\alpha) < r_\alpha$$

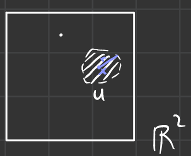
i.e, $\text{Ball}(x, r) \subset \text{Ball}(x_\alpha, r_\alpha)$. Since $\text{Ball}(x_\alpha, r_\alpha) \subset U$, we conclude $\text{Ball}(x, r) \subset U$. //

Rmk: The proof of this proposition (and its statement) only involve a notion of distance satisfying the triangular inequality. So the prop must be true for any set (not just $\mathbb{R}^n$) with such structure. Such a thing is called <u>metric space</u>.

Examples: $\forall$ integer $n \geq 0$, $\emptyset \subset \mathbb{R}^n$ is open.

Ex: $\mathbb{R}^n \subset \mathbb{R}^n$ is open

<u>Ex:</u>



is it open?

$\rightarrow$ If $U = \text{Ball}(x, 2) = \bigcup \text{Ball}(x, 2)$

<u>Ex:</u>