Spectra, operads, and ∞-categories

Notes from a PIMS summer school, July 2022

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(Drawing by Jeff Hicks)

Contents

A motivation in the context of our summer school

This is a summer school on what some people have come to call "Floer homotopy theory." Why do the topics of operads and spectra fit into such a summer school?

First, the Floer-type invariants that people find most concrete are certain groups. And most of these groups arise from chain complexes produced by analytic constructions. Modern algebra recognizes that chain complexes lie in the theory of modules over $\mathbb{Z}$, but $\mathbb{Z}$ is a particular ring. From higher-algebraic perspectives, a more fundamental ring is $\mathbb{S}$, the sphere spectrum. Thus a natural question – which was asked (and an affirmative answer hinted at) in Floer's original works[1] – is whether these invariants actually arise from well-chosen $\mathbb{S}$-modules, otherwise known as *spectra*. In other words, can we create Floer-type invariants that are spectra? The answer has been yes in many examples, and an open problem at the moment is a satisfactory construction of versions of Fukaya categories that are enriched over spectra (as opposed to chain complexes).

Second, one has always had more than mere groups in this game. Floer theory often gives rise to graded rings, for example. Sometimes these graded rings have more structure – they may be commutative, or they may have a Poisson structure; they may also have a degree 1 operator. One never likes a situation where structures appear in different places without a framework for organizing them all. *Operads* are precisely a language developed for organizing algebraic structures (though only for operations with one output, not multiple). I will indicate both the naturalness, and the clunkiness, of the classical notion of operads.

This is how the topics fit into the bigger mathematical world. I believe this summer school is meant to orient you in this landscape, which can look disjointed and confusing to a newcomer.

Not apparent from the names of these topics is another trend in mathematics. We have, in the last two decades, developed a robust and highly satisfactory theory of ∞-categories. For those who do not work in the field, the word "∞" is a mysterious one. The proliferation of this prefix can induce the same discomfort one feels when witnessing the explosion of "quantum" appearing in every math term possible. What's next, quantum ∞-proofs?

[1]See Figure .0.0.1

What is often under-appreciated is that the theory of ∞-categories, as developed first by André Joyal and later by Jacob Lurie, is actually rooted in the combinatorics of posets. Indeed, the discovery of ∞-categories might be phrased as the discovery that the collection of linear, finite posets contain enough structure to organize almost all homotopical phenomena one encounters in life. (This is under the assumption that all homotopical structures you want are rooted in some model for spaces and for associativity.)

I had to balance a lot in these lectures – set you up with the vocabulary to understand emerging research, explain how the homotopical ideas relate to the analytical ideas of Floer theory, and (this is a personal mission I have affixed, independent of requests of the organizers) catch some glimpses of the Wizard of Oz behind the ∞-curtain. I didn't quite pull off everything, but I hope that these notes will be a resource for the topics I did manage to write about.

The fact that the trajectory spaces are framed rather than only oriented suggests the following extension of the above program: If h^* denotes a general cohomology theory, then it should be possible to obtain $h^*(I(S))$ in a way similar to that above through an analysis of trajectory spaces. The chain complex would have to be replaced by $h^*(I(C)) = \bigoplus_{x \in C} h^{*-\mu(x)}$, and the δ-homomorphism would, in contrast to the singular case, depend on

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trajectory spaces of arbitrary dimension. For example, in the case of stable cohomotopy $h^* = \pi_s^*$, the contribution of every compact component of $\hat{M}(x,y)$ should be given by the element of π_s^* classifying its framed cobordism type. (In fact, the spaces $\hat{M}(x,y)$ undergo framed cobordisms under a change of the metric.) Of course, the higher dimensional components of $\hat{M}(x,y)$ do not have to be compact, but a reasonable modification of the program should lead to a spectral sequence converging to $h^*(I(S))$, as one would expect. This program is only of limited use for finite dimensional Morse theory, but might have applications to infinite dimensional cases.

FIGURE .0.0.1. An excerpt from Floer's paper. Andreas Floer, "Witten's complex and infinite-dimensional Morse theory." J. Differential Geom. 30(1): 207-221 (1989). DOI: 10.4310/jdg/1214443291.

A request for pictures and corrections

If you notice any images missing (I will indicate by blue font where I would love an image to be made), please feel free to submit an image – preferably drawn (on a tablet, on a sheet of paper, etc) – to incorporate. It would be lovely to have these images, and in case of redundancies, it would be lovely to have a gallery of images submitted by you.

And, of course, if you spot any typos, please feel free to e-mail Hiro to let him know.

Acknowledgments

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First and foremost, I would like to thank the students at the summer school for their questions; many answers made their way into these notes. The sense of joy and community you brought was fuel for what were two grueling weeks for this speaker. Thank you to the organizers of the summer school – Kristen Hendricks, Ailsa Keating, Robert Lipshitz, Liam Watson, and Ben Williams – and to Ruth Situma of PIMS for all her work in making the school possible.

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Week 1

LECTURE I

Spectra as "abelian groups" (abelianness, homotopically)

[1]

One can give many equivalent definitions of spectra. We give a definition that (we think) is most easily accessed and motivated for algebra: Spectra are topological spaces equipped with the data of being "abelian groups" in a homotopically natural sense.

This is the model of spectra as infinite loop spaces, also called Ω-spectra in much of the literature. (Definition I.2.0.1.)

WARNING I.0.0.1. Spectrum could mean the spectrum of a linear operator; it could mean the Zariski spectrum of a commutative ring; or it could mean the Balmer spectrum of a tensor-triangulated category. None of these is the notion of spectrum in these lectures.

See also Section I.8 for some discussion on what others might mean by an "abelian group" in homotopy theory.

I.1. Proving Abelianness

A topologist's favorite objects do not arise as abelian groups, but as topological spaces. How might a space aficionado incorporate "additivity" into their world?

I.1.1. Homotopy groups. There are probably two places you first encounter groups in an algebraic topology course: (Co)homology, and fundamental groups. The former is an artificial source, as one forces a coefficient group into the picture. In contrast, the fundamental group arises from a topological fact: Loops compose.

RECOLLECTION I.1.1.1. Fix a topological space X and a basepoint $x_0 \in X$. The fundamental group

$$\pi_1(X, x_0)$$

is defined to be the set of homotopy classes of continuous maps $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$.[2] The group multiplication is induced by concatenating paths.

[1]**Note.** *This lecture is nearly identical to the first lecture I gave at the MSRI lecture series on spectra in Spring 2019 (during the Derived Algebraic Geometry program).*

[2]We demand that the homotopies are homotopies through maps satisfying this condition.

NOTATION I.1.1.2. When X is path-connected, we will often leave the basepoint implicit and write $\pi_1(X)$.

EXAMPLE I.1.1.3. If $X = S^1$ is the circle, then $\pi_1(S^1) \cong \mathbb{Z}$, where the isomorphism is given by the winding number.

If $X = S^1 \vee S^1$ is the wedge sum of two circles, otherwise known as a figure eight, then $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$ is the free (non-abelian) group on two generators.

The fundamental group $\pi_1(X, x_0)$ is rarely abelian, so we surely aren't in any realm of "abelianness." But the higher homotopy groups are abelian.

DEFINITION I.1.1.4. Fix a basepoint $x_0 \in x$. Recall that

$$\pi_2(X, x_0)$$

is defined to be the set of homotopy classes of continuous paths

$$\gamma : [0,1] \times [0,1] \to X$$

such that γ sends the boundary of the square $[0,1] \times [0,1]$ to the basepoint x_0.

More generally,

$$\pi_n(X, x_0)$$

is the set of homotopy classes of maps from an n-dimensional cube to X, such that the boundary of the cube is sent to x_0. We call $\pi_n(X, x_0)$ the *nth homotopy group of* X (based at x_0).

Group multiplication $[\gamma_2][\gamma_1]$ is defined by choosing an embedding of two cubes into a single cube, and then defining a map which equals γ_1 and γ_2 on the two embedded cubes, but is constant with value x_0 outside the embedded cubes. (See Figure I.1.1.5.)

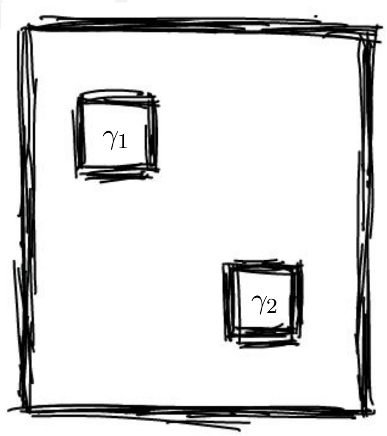

FIGURE I.1.1.5. A representative of the product $[\gamma_2][\gamma_1]$, induced by an embedding of two cubes into one.

I.1.2. Why are higher homotopy groups abelian? The proof that higher homotopy groups are abelian is called the *Eckmann-Hilton argument*, which exhibits a homotopy between the multiplications $[\gamma_2][\gamma_1]$ and $[\gamma_1][\gamma_2]$. For example, for π_2, one often draws a picture as follows:

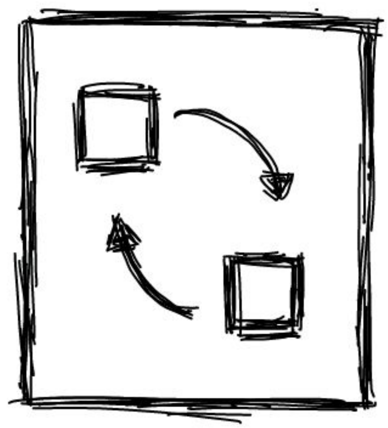

FIGURE I.1.2.1. A homotopy swapping the placement of two embedded cubes.

Because the multiplication with γ_1 and γ_2 swapped is homotopic to the original composition (via the drawn homotopy), we conclude $[\gamma_2][\gamma_1] = [\gamma_1][\gamma_2] \in \pi_2$.

This proof is worth examining. For example, what if we post-compose, or pre-compose, the homotopy of Figure I.1.2.1 with the homotopy given by one cube winding around the other cube?

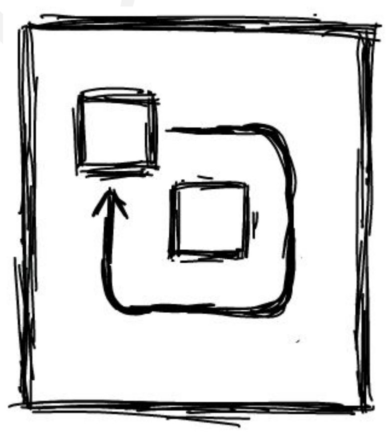

FIGURE I.1.2.2. A homotopy from a multiplication to itself.

The result is *another* homotopy showing $[\gamma_2][\gamma_1] = [\gamma_1][\gamma_2]$; and this homotopy is different from the previous one. Indeed, the number of times we wind around (the winding number) distinguishes the homotopies. In other words, although $[\gamma_2][\gamma_1]$ is homotopic to $[\gamma_1][\gamma_2]$, there are *inequivalent ways* in which they are homotopic.

Takeaway. The space of ways in which we witness π_2 being abelian is not trivial.

How about π_3? This is also abelian. And because we can move around in three dimensions now (as we homotope cubes past each other) the loop depicted in Figure I.1.2.2 can be made null-homotopic.

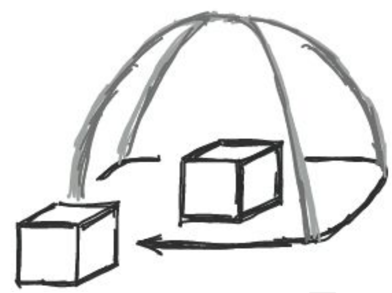

FIGURE I.1.2.3. A disk/dome exhibiting a null homotopy of the loop from Figure I.1.2.2.

So the winding number obstruction is trivializable in the Eckmann-Hilton argument for π_3.

But while we have depicted a northern hemisphere in Figure I.1.2.3 to trivialize a winding number, a friend (or nemesis) could have chosen a southern hemisphere. Thus the way to try and trivialize your proof of abelianness is non-canonical – to nullify the winding number, you have to make choices, and this choice is again non-trivial, because we see a non-contractible 2-sphere appear. So again, the collection of ways in which (one can prove that) π_3 is abelian forms a non-trivial space. Of course there is nothing special about $n = 2$ and $n = 3$. For general n, we witness a sphere of dimension $n - 1$ appear.

So even though each $\pi_n(X, x_0)$ is abelian for $n \geq 2$, Mother Nature does not make these groups canonically abelian.

What happens as n goes to infinity? We can transport the proof of the abelianness of π_n to a proof of π_{n+1} being abelian by (pictorially) taking direct products with closed intervals; in this way we get embeddings

$$S^{n-1} \subset S^n \subset \dots$$

of the spaces appearing when we try to prove π_n to be abelian. The key observation is that these non-trivial spaces of abelian-witnessing become simpler as we go to higher dimensions: $S^\infty = \varinjlim_n S^n$ is contractible.

So as n goes to infinity, we can articulate abelianness with only *contractible* ambiguity about how we want to express the fact that composition order doesn't matter. And to a homotopy theorist, contractibility is triviality. That is, we can witness an "abelianness" which is homotopically canonical.

Inspired by this vague idea, I would now like to make precise the notion of infinite loop spaces.

I.1.3. Remembering homotopies through loop spaces. I want to remember the space of ways in which blocks might move past each other. How? Well, we lost that space by modding out by homotopies. So let's not do that:

DEFINITION I.1.3.1 (ΩX, $\Omega^n X$.). Let X be a topological space, and choose a point $x_0 \in X$. We let

$$\Omega^n(X)$$

be the space[3] of continuous functions

$$\gamma : [0,1] \times \ldots \times [0,1] \to X$$

(from the n-dimensional cube to X) sending the boundary of the n-cube $[0,1]^n$ to the base point $x_0 \in X$. We call $\Omega^n X$ the *n-fold* based loop space of X.

When $n = 1$, the space

$$\Omega X$$

is the *based loop space* of X.

REMARK I.1.3.2. A more faithful notation might be $\Omega_{x_0}(X)$ or $\Omega^n_{x_0} X$, but we often ignore x_0 in the notation to avoid clutter.

Note that if X is path-connected, any two choices of x_0 yield homotopy equivalent loop spaces.

The following are elementary, but they will be used over and over again.

REMARK I.1.3.3. Note that $\Omega^n_{x_0}(X)$ has a natural basepoint given by the constant map γ_0 taking value x_0. Using this basepoint, we have $\Omega^{n+1}_{x_0}(X) \cong \Omega_{\gamma_0}(\Omega^n_{x_0}(X))$. Or, in less cluttered notation,

$$\Omega^{n+1} X \cong \Omega(\Omega^n X).$$

Thus one may interpret the notation Ω^n as $\Omega \circ \ldots \circ \Omega$.

NOTATION I.1.3.4. For any space X, we let $\pi_0(X)$ denote the set of path-connected components of X.

EXAMPLE I.1.3.5. In particular $\pi_0 \Omega^n(X) \cong \pi_n(X, x_0)$. So one can think of $\pi_n(X, x_0)$ as the group we obtain by beginning with a beautiful space called $\Omega^n_{x_0}(X)$ and destroying it by only remembering its connected components. Flipping this observation, we think of $\Omega^n X$ as a space that allows us to *remember* how maps out of spheres can be composed.

[3]This set can be given the compact-open topology. A path in this space from γ to γ' is precisely a homotopy between γ and γ' respecting the basepoints. Even better is to endow the space with the *k-ified* topology, to render the topology compactly generated.

Because $\pi_1 X$ is not abelian, we should think of ΩX as not at all abelian. The two-fold loop space $\Omega^2 X$ is "more abelian," because we can arrange for the order of composition to not matter, up to homotopy. As we discussed in the previous section, the intuition is that $\Omega^n X$ is "more abelian" the higher n is, because the space witnessing abelianness is roughly equivalent to S^{n-1}.

Finally tracing through this line of thought, we arrive at the following conclusion: Something which behaves like an n-fold loop space for arbitrary large n is "canonically" abelian, because the space of ways in which we witness two elements commuting is contractible.

We may at last answer the question of what we mean by an "abelian group" structure on a space X. To give a space $X = X_0$ a "group structure," one can exhibit an equivalence with ΩX_1 for some space X_1. Then X_0 inherits the loop composition and becomes a group, though not a very abelian one. To endow this multiplication with abelianness, we can exhibit an equivalence $X_0 \simeq \Omega X_1 \simeq \Omega^2 X_2$, where the last equivalence is induced by choosing $X_1 \simeq \Omega X_2$ and applying Ω. (This ensures that the original group multiplication inherited from X_1 is compatible with the one inherited from X_2.) And so forth. We must specify an infinite chain of equivalences

$$X_0 \simeq \ldots \simeq \Omega^n X_n \simeq \ldots.$$

Note that abelianness is now a far cry from being a property of a single group structure; it is the specification of data.

Remark I.1.3.6. Recall that, when we mod out by homotopy, the set $\pi_n(X, x_0)$ has a single multiplication map. In contrast, there are *many* natural maps

$$\Omega^n X \times \Omega^n X \to \Omega^n X.$$

(There is roughly one such map for every way in which one can embed two cubes into a larger one.) While these choices are all homotopic for $n \geq 2$, as we learned in the previous section, the space of ways to embed cubes into a bigger cube is itself an interesting space.

Remark I.1.3.7. We have also not discussed associativity. For example, there is an even more complicated space of ways in which we can embed k many cubes to realize maps

$$\Omega^n X \times \ldots \times \Omega^n X \to \Omega^n X$$

where the product is taken k times. The vocabulary of *operads* is meant to codify in what sense we can think of coherent algebraic structures (such as associativity and commutativity). This is of course a preview of a later lecture. The interested reader can look up "the little n-cubes operad" or "the $\mathbb{E}_n$ operad." The space $\Omega^n X$ is an algebra for this operad.

I.2. Spectra

We have reached the following guess at an "abelian group"-like structure for spaces:

Definition I.2.0.1 (Spectra). A *spectrum* X is the data of

(1) Topological spaces $X_0, X_1, \ldots$ together with chosen points $x_i \in X_i$ for each X_i, and
(2) Homotopy equivalences[4] $X_i \xrightarrow{\sim} \Omega X_{i+1}$ for every i. Here, ΩX_i is the based loop space taken at the basepoint $x_i \in X_i$ from above.

Warning I.2.0.2. There is no useful definition of "$\Omega^\infty X$" for a topological space X; indeed, if one tried to define such a thing as the space of maps $\gamma : S^\infty \to X$ sending a basepoint of S^∞ to x_0, this space would be contractible, and would have no interesting structures. In contrast, the data defining a spectrum can, in practice, often form a subtle collection.

Let me mention now some notation that can be confusing. In later chapters, we will write $\Omega^\infty Y$ will to denote the space associated to a spectrum Y. It will turn out that $\Omega^\infty Y = Y_0$, the 0th space in the above sequence.

Remark I.2.0.3. Note that if one begins with a space X_0, there may be many inequivalent ways to produce a spectrum whose 0th space is X_0. For example, even in the case of $X_0 = \mathbb{Z}$, while an obvious first choice is $X_1 = S^1$, one could also take $X_1 = S^1 \times K$ for any discrete space K with an abelian group structure; then by choosing a basepoint $x_1 \in X_1$, we obtain still that $\Omega X_1 \simeq X_0$. There are similar ambiguities in choosing X_n for higher n.

Remark I.2.0.4. In general, most spaces do not arise as X_0 of a spectrum. One obvious obstruction is whether $\pi_1 X_0$ is abelian. This is not the only one.

Definition I.2.0.5. A morphism of spectra $X \to Y$ is the data of continuous maps

$$f_i : X_i \to Y_i$$

respecting basepoints, and of homotopies

$$H_i : X_i \times [0,1] \to \Omega Y_{i+1}$$

rendering the diagram

$$\begin{tikzcd}
X_i \arrow[r, "\sim"] \arrow[d] & \Omega X_{i+1} \arrow[d] \\
Y_i \arrow[r, "\sim"] & \Omega Y_{i+1}
\end{tikzcd}$$

homotopy commutative.

We let

$$\hom_{\mathsf{Spectra}}(X, Y)$$

[4] It is very important to think of this arrow as pointing in the direction indicated. Indeed, the data of an arrow in this direction is only equivalent to the data of "an arrow in the other direction, plus homotopies exhibiting this other arrow as a homotopy inverse." Put another way, the space of homotopy inverses to a given map is not contractible; only the space of homotopy- inverses-equipped-with-data-realizing-their-homotopy-inverseness is contractible.

denote the space of maps of spectra.

Remark I.2.0.6. Informally, the data of a map f allows us to conclude the map $f_0 : X_0 \to Y_0$ is a map of groups (since H_0 exhibits it as homotopic to a map of loop spaces Ωf_1). The other homotopy-commutative squares encode data making the group map respect the abelianness inherited from being higher loop spaces.

Remark I.2.0.7 (Composition). The astute reader will wonder how to compose two morphisms of spectra, as each morphism involves data of homotopies. In particular, if a homotopy is parametrized by an interval of the form $[0, 1]$, do we want a composition of two homotopies to be parametrized over $[0, 2]$? One classical work-around is to use the Moore path space model for concatenating homotopies (by declaring that in general by a homotopy we mean something parametrized over $[0, t]$ for whatever $t \geq 0$).

Another classical work-around is to invoke the language of operads (in the sense of Peter May), and model spectra as a category over an A_∞-operad. This is the same strategy that allowed one to articulate some of the “associative up to homotopy” structures enjoyed by loop spaces.

A third work-around, which we will not discuss in detail, is to create an ∞-category of spectra, where in we replace the data of composition and coherence by simply declaring what “coherent diagrams” mean for spectra. This would be my preferred method. It should be said that any of the previous two methods would indeed make it rather difficult to articulate the symmetric monoidal structure of smash product.

Even in light of this remark, you should know that a composition of two maps of spectra, f, f' involves the usual composition $f' \circ f$ and a concatenation of $f'(H)$ with H'. Any two such concatenations yield homotopic data, so we do not care which you choose. The mistake (or arduous path) would be to insist on “the” composition. The healthy attitude is to treat any two reasonable candidates for a particular composition to be acceptable.

Definition I.2.0.8 (Equivalence of spectra). A map $f : X \to Y$ is called an *equivalence* if there exists a map $g : Y \to X$ for which there exists homotopies $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$. (These homotopies are through maps of spectra; i.e., they are paths in the mapping spaces $\hom_{\mathsf{Spectra}}(X, X)$ and $\hom_{\mathsf{Spectra}}(Y, Y)$.)

Remark I.2.0.9. There is an algebraic way to characterize equivalences by computing the induced map on homotopy groups of spectra, similar to the classical Whitehead theorem for spaces. See Exercise II.18.

I.3. How do you compute addition?

In principle, the first lecture is over. In the following sections, me answer some common questions a reader may have. The contents of the following sections will be assumed in later lectures.

I motivated a spectrum as like an abelian group for homotopy theorists. So if I exhibit X_0 as the 0th space of some spectrum X, what is the "addition?"

By the homotopy equivalence $X_0 \to \Omega X_1$, I can think of two elements of X_0 as two loops in X_1. Concatenate those loops. The resulting loop is the sum of the two elements.

Of course, everything is up to homotopy. As you know, there are many possible ways to concatenate two loops (corresponding to the ways in which I allocate time intervals inside $[0, 1]$, for example) so there isn't a single well-defined sum/concatenation. And the concatenated loop may not be in the image of $X_0 \to \Omega X_1$; but because f_0 is a homotopy equivalence, one can identify points in X_0 that deserve to be called the sum up to homotopical data.

One could think of the other maps $X_i \to \Omega X_{i+1}$ as existing purely to exhibit the "commutativity" data, but of course you can interpret addition through them as well. The map $X_0 \xrightarrow{\sim} \Omega^n X_n$ allows you to think of each point in X_0 as some n-sphere in X_n, and you can concatenate/compose those. Because we also have an equivalence $\Omega X_1 \to \Omega^n X_n$ which is a loop map, the addition operations in these paragraphs are all equivalent up to only contractible ambiguities as $n \to \infty$.

I.4. Example: The integers

Surely, $\mathbb{Z}$ ought to be an abelian group for a homotopy theorist; after all, that $a+b$ *equals* $b+a$ is a pretty canonical way to exhibit commutativity of an operation.

But to fit things into our definition from the previous section, we must be able to exhibit $X_0 = \mathbb{Z}$ as an infinite loop space. So can we find a space X_1 so that $\mathbb{Z} \simeq \Omega X_1$? Then we must also exhibit $X_1 \simeq \Omega X_2$ for some space X_2, and so forth. One already sees something non-trivial about this.

EXAMPLE I.4.0.1. The circle S^1 is a space whose fundamental group is $\mathbb{Z}$, and whose higher homotopy groups are zero. (One can see this because the universal cover of S^1 is $\mathbb{R}$, which is contractible, hence has no homotopy groups; moreover, covering maps induce isomorphisms on all π_k for $k \geq 2$.) So our first stab at X_1 is S^1; indeed, you can try convincing yourself that ΩS^1 is homotopy equivalent to the discrete space (i.e., set) $\mathbb{Z}$, and the map $\Omega S^1 \to \mathbb{Z}$ exhibiting this equivalence is given by the winding number.

But what space has S^1 as its space of based loops? Recall from Exercise I.10 that Ω shifts homotopy groups. So to exhibit $\mathbb{Z}$ as an infinite loop space, we need to find spaces X_i such that the ith homotopy group of X_i is exactly $\mathbb{Z}$, and all higher homotopy groups of X_i are zero. Moreover, let us demand all lower homotopy groups also vanish so as to minimize the chance of making non-canonical choices. (See Remark I.2.0.3.)

Such spaces – with only one non-vanishing homotopy group – have a long history in topology.

Definition I.4.0.2. An *Eilenberg-Maclane space* is a topological space Y whose homotopy groups $\pi_n(Y)$ are non-zero for at most one value of n. If $\pi_n(Y) \cong A \neq 0$, we often denote Y by the symbol

$$K(A, n).$$

Remark I.4.0.3. It is common to denote a choice of abelian group by π rather than by A, so you will often hear in conversation or in writing the name "$K(\pi, n)$" to refer to an Eilenberg-MacLane space.

It is uncommon to use the notation $K(0, d)$ for the (contractible) Eilenberg-MacLane space, but you would not be excommunicated for doing so.

Remark I.4.0.4 (Existence of Eilenberg-Maclane spaces). If somebody asks you to construct a chain complex with certain (co)homology groups, you'd do so easily. Just take a chain complex with zero differential, and prescribed groups in each degree. One can in fact make *free* chain complexes with prescribed cohomology – if we assume the cohomology groups are bounded in some direction, one can easily do so inductively.

Likewise, for a group P_1, it is easy to construct a connected space A_1 with $\pi_1 A_1 \cong P_1$; given a generators and relation presentation of P_1, one can simply wedge together a collection of circles, and attach disks to kill off the required relations.

This produces a space A_1 with the desired $\pi_1 \cong P_1$. But depending on your presentation, you may have introduced some π_2, and in turn some π_3 and so forth. (Recall that, aside from the circle, every sphere admits higher-dimensional homotopy groups – for example, $\pi_3(S^2) \cong \mathbb{Z}$.)

Regardless, given an abelian group P_i, we attach i-dimensional spheres to A_{i-1} to generate as necessary, and kill unwanted elements of (or impose relations on) π_i by attaching $(i+1)$-dimensional disks. This results in a space A_i with $\pi_i(A_i) \cong P_i$, while the lower-dimensional homotopy groups are unchanged. We proceed by attaching spheres and killing-disks inductively on i to obtain a space with the desired homotopy groups.

In particular, this shows we can construct Eilenberg-Maclane spaces (by setting all but one P_i to equal zero), though after making many arbitrary choices.

Because of our choices, it is not clear how homotopy equivalent our end results are. Certainly for arbitrary prescriptions of homotopy groups P_i, we have no hope for an equivalence. But for Eilenberg-MacLane spaces, we can guarantee it.

Remark I.4.0.5 (Uniqueness). Fix an Abelian group A and a non-negative integer $n \geq 0$. Two choices of $Y \simeq K(A, n)$ may have different homeomorphism types, but a classical theorem states that reasonable Eilenberg-MacLane spaces are unique up to homotopy equivalence. For example, both S^1 and $\mathbb{C} \setminus \{0\}$ are examples of $K(\mathbb{Z}, 1)$, and these are definitely not homeomorphic, but they are homotopy equivalent.

Here is one way to prove they are unique up to homotopy equivalence: In a category of reasonable spaces, one proves that any Eilenberg-Maclane space represents the functor sending a (reasonable) space X to the cohomology group $H^n(X; A)$. You can prove this not in the category of topological spaces, but its *homotopy category*, where $\hom(X, Y)$ is the set of homotopy classes of continuous maps from X to Y. Then by the Yoneda Lemma, any two Y, Y' representing $H^n(-; A)$ must be isomorphic in the homotopy category, meaning they are homotopy equivalent.

By uniqueness (up to homotopy equivalence) of Eilenberg-MacLane spaces, we formally conclude that there exist *unique* spaces (up to homotopy equivalence) $X_n = K(\mathbb{Z}, n)$ and homotopy equivalences such that

$$\mathbb{Z} = X_0 \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \dots.$$

Remark I.4.0.6. One can also give concrete models of each X_i. For example, $X_1 = S^1$ and $X_2 = \mathbb{C}P^\infty$, while more generally, one can define X_n to be the configuration space of finitely many points on S^n labeled by integers. (When points collide, the integer-labels add, and if a point is labeled by 0, it may disappear. S^n also has a basepoint into which any configuration point may disappear.) This last assertion is a consequence of the Dold-Thom theorem.

At the same time, one can make use of many objects through the functors they represent (without getting hands dirty with concrete models). The $K(A, n)$ represent the nth cohomology groups with coefficients in A, so one can often make use of them without the above models.

So we have finally exhibited the most basic abelian group, the integers, as an infinite loop space.

In fact, the above shows that for any abelian group A and for any integer $n \geq 0$, the space $K(A, n)$ may be exhibited as the 0th space of a spectrum, by setting

$$X_0 = K(A, n), \qquad X_1 = K(A, n+1), \qquad X_2 = K(A, n+2), \dots.$$

Definition I.4.0.7. Fix an abelian group A. Then the spectrum HA whose ith space is given by $K(a, i)$ is called the *Eilenberg-Maclane spectrum* associated to A.

Remark I.4.0.8. An astute reader may have noted that, in principle, there may be other ways to make $\mathbb{Z} = X_0$ into the 0th space of a spectrum.

As an example the space X_1 could have been chosen to have non-trivial π_0, and this would not affect the fact that $\Omega X_1 \simeq \mathbb{Z}$. If one had taken $X_1 = \mathbb{Z} \times K(A, 1) \simeq \mathbb{Z} \times S^1$, one could construct a new spectrum with $X_i = K(A, i-1) \times K(A, i)$. The end result is a spectrum with $\pi_0 = \pi_{-1} = \mathbb{Z}$, and we will later see that this is the wedge sum/direct sum $H\mathbb{Z} \oplus H\mathbb{Z}[-1]$.

The spectrum $H\mathbb{Z}$ is the unique *connective* spectrum whose 0th space is $\mathbb{Z}$. In fact, the Eilenberg-Maclane spectrum HA is the unique spectrum whose homotopy groups are trivial except in degree 0, where $\pi_0 \cong A$. See Exercise I.20.

I.5. Homotopy groups and shifts

Now that we have a definition of spectrum, we discuss homotopy groups and shifts of spectra.

Definition I.5.0.1 (Homotopy groups of a spectrum). Let X be a spectrum, so we are given spaces X_i and homotopy equivalences $X_i \simeq \Omega X_{i+1}$ for $i \geq 0$. Then the nth homotopy group of X is defined to be

$$\pi_n(X) := \pi_{n+i}(X_i)$$

where if n is negative, the righthand side makes sense so long as $n + i \geq 0$. For example, $\pi_{-1}(X)$ is the 0th homotopy group of X_1.

Remark I.5.0.2. The data of the weak equivalences give us group isomorphisms

$$\pi_{n+i}(X_i) \cong \pi_n(\Omega_i X_i) \cong \pi_n(X_0)$$

when $n \geq 0$. We also have that $\pi_{-n}(X) = \pi_0(X_n)$.

Definition I.5.0.3 (Connective spectra). A spectrum is called *connective* if all its negative homotopy groups vanish.

Example I.5.0.4. Let A be an abelian group. Because we chose/defined an Eilenberg-Maclane spectrum HA to have $X_i \simeq K(A, i)$, the homotopy groups are

$$\pi_n(HA) \cong \begin{cases} A & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Eilenberg-Maclane spectra associated to abelian groups are connective.

I.6. Chain complexes (not covered in spoken lecture)

It turns out we can also convert any chain complex into a spectrum. We will denote this functor from chain complexes to spectra by H, so a chain complex A is sent to a spectrum HA. (This generalizes the construction from the previous section, where A was an abelian group.)

Moreover, this construction enjoys the property that $H^k(A) \cong \pi_{-k}(HA)$. That is, cohomology groups of A are given by homotopy groups of HA. Note that if we were to have no notion of *negative* homotopy groups, one would not be happy with HA. After all, if A is the hom complex of two projective resolutions of some modules, it is only the *positive* cohomology groups that recover interesting invariants such as Ext groups.

If A is a chain complex, we can form its shifts $A[n]$ defined by $(A[n])^i = A^{n+i}$, regardless of the sign of n. This shifting has the pleasant feature that $A[1][-1] \simeq A$, and that $[n]$ obviously shifts the cohomology groups of A by n.

Likewise, spectra have a shift operation. Let X be a spectrum. Let us observe that our $\mathbb{Z}_{\geq 0}$-indexed sequence of spaces

$$X_0, X_1, \ldots$$

can actually be extended to a $\mathbb{Z}$-indexed sequence

$$\ldots, X_{-1}, X_0, X_1, \ldots, \tag{I.6.1}$$

by setting $X_{-i} := \Omega^i X_0$. Then for any $i \in \mathbb{Z}$, we automatically have the condition $X_i \simeq \Omega X_{i+1}$. (As this comment makes clear, a spectrum in some sense only cares about the "positive tail" of its sequence of spaces.) At this point the shift operation is manifest:

Definition I.6.0.1. Let X be a spectrum. Then $X[1]$ is the spectrum whose ith space is given by $(X[1])_i = X_{i+1}$. On the other hand, $X[-1]$ is the spectrum whose ith space is given by $(X[-1])_i = \Omega X_i$, or by X_{i-1} using the convention from (I.6.1).

If $f : X \to Y$ is a map of spectra, we have the obvious induced maps $f[\pm 1] : X[\pm 1] \to Y[\pm 1]$.

The shift functors indeed shift homotopy groups:

Proposition I.6.0.2. We have natural isomorphisms

$$\pi_n(X) \cong \pi_{n+1}(X[1]) \cong \pi_{n-1}(X[-1]).$$

Remark I.6.0.3. For spectra and for chain complexes, we will later see that this shift operation is not merely a *construction* of an operation, but satisfies a universal property; they are homotopy pushouts and pullbacks along 0 maps.

Moreover, if H is the functor sending a chain complex A to a spectrum HA, it turns out $H(A[1]) \simeq (HA)[1]$. That is, there is a natural shift operation on spectra, just as there is for chain complexes, and H is compatible with this shift operation.

I.7. Reading (not covered in spoken lecture): The history of definitions of spectra

The development of stable homotopy theory – that is, the field of math studying spectra – is a beautiful example of the non-uniformity of progress.

Here is an analogy I learned from a talk of Mori[5]. Mathematical invariants are like a cubist painting of the objects we seek to describe – we cannot see the object itself, but we aggregate facets of it. (See Figure I.7.0.1.)

I think historical attempts at describing spectra captured facets of spectra. But there was difficulty in pinning down a usable and intuitive definition. This difficulty is a heavy piece of evidence in favor of people who believe math is about good definitions.

[5] I heard this specifically at Shigefumi Mori's talk at the June 2016 Benjamin Peirce Centennial Conference at Harvard, "Rational Curves on Algebraic Varieties - an encounter at Harvard and its later development."

FIGURE I.7.0.1. Violin and Candlestick, 1910 by Georges Braque

Even today, a Wikipedia article on spectra will give a definition of a spectrum that is what I call a prespectrum in Exercise I.19. Many people believe that prespectra are a more natural definition – for historical reasons, and because many geometric examples arise as prespectra. My personal opinion is that Ω-spectra *are* spectra, full stop; but let's not get into arguments here.

Without getting into the history too much, shown in Figure I.7.0.2 is a page from J. Frank Adams's "Stable Homotopy Theory," published in 1964 by Springer-Verlag in the Lecture Notes in Mathematics series, based on lectures given at UC Berkeley in 1961.

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justifiable objects, but they don't exist inside S-theory.

I want to go ahead and construct a stable category. Now I should warn you that the proper definitions here are still a matter for much pleasurable argumentation among the experts. The debate is between two attitudes, which I'll personify as the tortoise and the hare. The hare is an idealist: his preferred position is one of elegant and all embracing generality. He wants to build a new heaven and a new earth and no half-measures. If he had to construct the real numbers he'd begin by taking <u>all</u> sequences of rationals, and only introduce that tiresome condition about convergence when he was absolutely forced to.

The tortoise, on the other hand, takes a much more restrictive view. He says that his modest aim is to make a cleaner statement of known theorems, and he'd like to put a lot of restrictions on his stable objects so as to be sure that his category has all the good properties he may need. Of course, the tortoise tends to put on more restrictions than are necessary, but the truth is that the restrictions give him confidence.

You can decide which side you're on by contemplating the Spanier-Whitehead dual of an Eilenberg-MacLane object. This is a "complex" with "cells" in all stable dimensions from $-\infty$ to $-n$. According to the hare, Eilenberg-

FIGURE I.7.0.2. A page from J. Frank Adams's "Stable Homotopy Theory," published in 1964 by Springer-Verlag in the Lecture Notes in Mathematics series, based on lectures given at UC Berkeley in 1961.

Of course, whether an approach is hare-brained or tortoise-brained is only subjectively appreciable, and opinions change over time. What feels abstract and general one day can, with practice, feel like the most concrete manifestation of an idea.

I will most likely be closer to a hare than a tortoise in these lectures, but I hope to empower you with enough intuition that you could at least identify whether a tortoise is headed in the right direction with their constructions.

Here is another quote from Adams, many years later:

> "The important thing is to know that there is a good category of spectra, and not to insist on any one choice of details for its construction. In fact there are alternative ways of setting up the details; they all lead to the good category, but for some particular application one may have an advantage over another. Let us keep our options open."

This quote is from J. Frank Adams's "Infinite Loop Spaces" (Chapter 1, Section 3). The book was based on the Hermann Weyl Lectures at the IAS, given in Spring 1975. The book (part of the orange Annals of Math studies series published by Princeton University Press) was published in 1978.

You can see that for at least 17 years – even as the problem of building concrete models of spectra progressed – one felt one should be open-minded about which models to use in which situations. At the same time, it was possible to have an opinion that there was "the" good definition (in Adams's own quote about keeping options open!).

Let me explain why there is no contradiction here. There are often different ways of defining a desired category, and these different-looking ways are often equivalent. Indeed, major theorems are about equivalences of things that look very different.[6] Adams is saying that, depending on the context, one side of the equivalence may be a more convenient model than another.

But then the problem arises: How do you characterize this particular equivalence class? "All models of spectra are equivalent" is a frustrating statement if we don't have a universal characterization of spectra. As an analogy, "every Cauchy sequence of rational numbers converging to e is equivalent" is so tautological that it borders on meaningless without having some characterization of e itself that does not depend on a particular choice of Cauchy sequence to begin with.

My personal opinion is that there is indeed one "good" way to set everything up, based on formal properties about symmetric monoidal stable ∞-categories: $\mathsf{Spectra}$ is the unit in the ∞-category of presentable stable ∞-categories. (This is a fact that must be true regardless of what model of higher category theory you take.) But this opinion is rooted in dirty work; so what I present in these lectures is a middle ground that avoids the details

[6] For example, that $D^bCoh(\mathbb{P}^1)$ is a derived category of quiver representations. Or that the K theory of finite sets is the sphere spectrum.

of the set-up. (See Section II.11.) I aim to present concrete models where they might help the reader, and abstract properties/philosophies where they help in justifying concrete models or in advancing the theory.

REMARK I.7.0.3. There are definitions of spectra like diagram spectra and symmetric spectra, where a spectrum is not only defined as some collection of spaces, but this collection is indexed by objects with more and more symmetries, and often with actions of these symmetries. Let me give one motivation as to why such things were created: At the end of the day, it was all to create a good model of the smash product (see next lecture). It is an interesting phenomenon. To encapsulate the idea of tensoring objects in a nice way, these models shoved the symmetries and structures of tensor products into definitions of the objects themselves. This certainly resolved some issues, but one cannot help but wonder if the desire for a smash product in the end obfuscated the definitions of the objects themselves.

Later in these lectures when I talk about operads, we will again encounter a shortcoming – the issues of coherences are shoved into particular regions of the definitions, but like attempts to smoothen an infinite and wrinkled carpet, one only serves to transport the bumps to another place. What we need is a robust framework for dealing with homotopy coherences to begin with.

So in fact, what has revolutionized the field in the last 16 years is not a new definition of spectra, but a new model for higher category theory – that of ∞-categories, due to Joyal and Lurie.

REMARK I.7.0.4. Just one more remark about this, again by historical analogy. The old definition of manifold was as a subset of Euclidean space satisfying some properties (smooth and locally Euclidean).[7] This is a frustrating definition because S^1 with its standard embedding is no different from S^1 with another embedding, yet anytime someone comes up with their own preferred model of the circle, you have to write down a darn diffeomorphism between them.

A newer definition defines a manifold as a space with some universal choice of atlas. This notion of "maximal atlas" (and perhaps, the underlying idea of transition charts) was a genuinely useful idea, and we see in hindsight how the old definition fits in: A choice of embedding gives a choice of atlas of a manifold by intersecting the manifold with open balls of the Euclidean space in which the manifold is embedded.

It is an intellectual leap to first think of something as defined using equations, then to think of an abstract way to define a smooth structure via a maximal atlas. Likewise, it is a leap to first think of the category of spectra as defined by specifying what objects and morphisms are, then to think of spectra as characterized by a universal property among all ∞-categories.

[7]Indeed, if you read some old writings of even the greats, they define a manifold as an object *defined* by a collection of smooth and transverse equations in $\mathbb{R}^n$. You can see the analogy with affine varieties.

Regardless of this beautiful story, I've in parallel tried to present first things first. So let's keep getting a feel not just for universal properties (like Theorem II.6.0.1), but what these universal things are actually supposed to be.

I.8. Abelian groups in spaces (not covered in lecture)

I should warn you that there are plenty of well-respected topologists who would reject my philosophy that "spectra are abelian groups." In fact, many people think of the homotopical notion of abelian groups in spaces as "spectra with vanishing negative homotopy groups" (i.e., connective spectra).

There is a sense in which both philosophies are correct. For example, if you formulate what you mean by an "abelian group object in the ∞-category of spaces," you will indeed come upon the answer of spectra with vanishing negative homotopy groups.

But this is not what I've formulated. I did not ask, at the outset, what the abelian group objects are in the ∞-category of spaces. I asked what I would invent if I followed my nose, pursuing what it means to impose a homotopical "abelian group structure" on a space. I inevitably arrived at spectra, and no intuition about abelian groups demanded along the way that "the nth space X_n should have no homotopy groups in degrees $\leq n-1$."

The distinction between these two perspectives is almost the point: The notion of "abelian group object" depends on a classical notion of abelian group. The approach in these notes arrives at a new and natural notion, suited for spaces, without relying on this classical notion.

Finally, let me also give a big warning. There is a difference between an abelian group in the ∞-category of spaces, and an abelian group in the category of spaces. (Read that again if you need to; see also Exercise II.20.) The latter, of course, are just topological abelian groups, and one can prove using the Dold-Kan correspondence that such things (when considered as spectra) are always direct products of Eilenberg-MacLane spectra. One can further prove, then, that this actually just recovers the homotopy theory of chain complexes over the integers. I promise that the theory of spectra is far more intricate than the theory of chain complexes.

Exercises about sets and spaces

I.9. Suspension and based loops

(a) Let X, Y, Z be sets. Exhibit a bijection
$$\hom(X \times Y, Z) \cong \hom(X, \hom(Y, Z))$$
where hom stands for the set of functions, and $X \times Y$ is the usual direct product.

(b) Suppose you have functions $f : X \to X', g : Y \to Y', h : Z' \to Z$. Show that the diagram below commutes:
$$\begin{tikzcd}
\hom(X \times Y, Z) \arrow[r, "\cong"] & \hom(X, \hom(Y, Z)) \\
\hom(X' \times Y', Z') \arrow[r, "\cong"] \arrow[u, "(f^* \times g^*)h_*"] & \hom(X', \hom(Y', Z')) \arrow[u, "(f^* \times g^*)h_*"']
\end{tikzcd}$$

(c) Now suppose that X, Y, Z are *pointed* sets, so we have chosen elements x_0, y_0, z_0 in X, Y, Z respectively. Exhibit a bijection
$$\hom_*(X \wedge Y, Z) \cong \hom_*(X, \hom_*(Y, Z))$$
where now $\hom_*$ stands for functions of pointed sets. (A function is pointed it if respects the chosen base points.) This is the "tensor hom" adjunction for pointed spaces, and $\wedge$ plays the role of tensor product. Concretely, the functor $-\wedge Y$ is a left adjoint to the functor $\hom_*(Y, -)$.

Remark I.9.0.1. Now suppose X, Y, Z are "nice" spaces with nice choses of basepoint. It is a fact that one can put a nice topology on the space of continuous functions between nice spaces so that all of the above bijections are homeomorphisms between mapping spaces. (A thorough discussion would involve the compact-open topology on hom-spaces, well-pointedness, and the k-ification of direct product of two spaces.)

(d) Now letting $Y = S^1$, and taking X, Z to be nice spaces with nice basepoints, exhibit a natural homeomorphism
$$\hom_*(\Sigma X, Z) \cong \hom_*(X, \Omega Z). \tag{I.9.1}$$
This is the suspension-loops adjunction. It literally states that a (pointed) map from the (reduced) suspension of X to Z should be thought of as a family of loops in Z indexed by X, and vice versa.

Remark I.9.0.2. Now-a-days, most people utilizing spectra do not care that (I.9.1) is a homeomorphism, and we only care that it is a homotopy equivalence. Indeed, though we gave point-set models of Σ and Ω, there are other ways to characterize Σ and Ω that only preserve these mapping spaces up to homotopy equivalence.

I.10. Based loops shift π_n, but not invertibly.

(a) For any pointed space X, and for any integer $k \geq 1$, prove we have a natural isomorphism of homotopy groups

$$\pi_k X \cong \pi_{k-1}\Omega X.$$

Remark I.10.0.1. That is, the Ω operation "shifts" homotopy groups. This is similar to the way that the shift operation for chain complexes shifts homology groups, with the key difference that Ω is not invertible.

(b) Let X be a topological space. Show that $\Omega\Sigma\Omega\Sigma X$ and $\Omega^2\Sigma^2 X$ are not homotopy equivalent. So, $(\Omega\Sigma)^k$ and $\Omega^k\Sigma^k$ are not equivalent functors.

(Hint: Try $X = S^0$. It may help to know that S^2 has at least some $\pi_k, k \geq 3$ which is non-zero. For example, $\pi_3(S^2) \cong \mathbb{Z}$. It may also help to know that the universal cover of a graph – i.e., of a 1-dimensional CW complex – is contractible, or that graphs have no higher homotopy groups.)

I.11. Composing loops versus pinching circles

We started off the exploration of "groups in homotopy theory" by observing that ΩX looks like a group – concatenating loops is the group operation, and up to homotopy, every loop has an inverse. Writing

$$\Omega X = \hom_*(S^1, X)$$

as the space of pointed continuous maps from the circle to X, one might wonder if the group structure on ΩX arises from another structure on S^1 itself.

(1) Given two pointed spaces A and B, their wedge sum $A \vee B$ is defined to be

$$A \vee B := (A \coprod B)/(a_0 \sim b_0).$$

In words, the wedge sum is the union $A \cup B$, where A is glued to B along the basepoints. Exhibit a natural homeomorphism of pointed mapping spaces

$$\hom_*(A \vee B, C) \cong \hom_*(A, C) \times \hom_*(B, C).$$

(Categorically, this says that $A \vee B$ is the coproduct in the category of pointed spaces. In words, to give a continuous pointed map from $A \vee B$ is the same as giving two continuous pointed maps – one from A, and another from B.)

(2) For any pointed space W, consider its reduced suspension $\Sigma W = S^1 \wedge W$. It turns out there exists a natural "pinch" map

$$\Sigma W \to \Sigma W \vee \Sigma W$$

where the target is the wedge sum. In light of the previous exercise, we thus obtain maps

$$\hom_*(\Sigma W, X) \times \hom_*(\Sigma W, X) \to \hom_*(\Sigma W, X).$$

By considering the case $W = S^1$, convince yourself that there is actually an interval's worth of pinch maps (roughly, the interval's worth of loop compositions) inducing the composition maps of ΩX.

(3) Why do you expect – for any W – the space $\hom_*(\Sigma W, X)$ to have a group structure? (You may identify this space with an appropriate based loop space, if you like.)

(4) Just to verify that your pinch maps are correct, verify that each pinch map has the effect

$$\widetilde{H}_k(\Sigma W) \to \widetilde{H}_k(\Sigma W) \oplus \widetilde{H}_k(\Sigma W), \qquad A \mapsto A \oplus A$$

on reduced homology.

I.12. Smash product for sets and spaces

I.12.1. Free and forget (for abelian groups). (This may be helpful for next lecture). Let X be a set. We let $\mathbb{Z}X$ denote the free abelian group generated by X.

Remark I.12.1.1. You can concretely model $\mathbb{Z}X$ as the direct sum $\oplus_X \mathbb{Z} = \mathbb{Z}^{\oplus X}$ of copies of integers, each summand indexed by an element of X. Let me remind you that if X is infinite, then $\oplus_X \mathbb{Z}$ is the set of X-tuples of integers, where only finitely many elements of the tuple are non-zero. Equivalently, $\mathbb{Z}^{\oplus X}$ is the set of all functions from X to $\mathbb{Z}$ for which all but finitely many elements of X are sent to 0.

Note that if $f : X \to X'$ is any function, one has an induced map of abelian groups (i.e., a group homomorphism) $\mathbb{Z}f : \mathbb{Z}X \to \mathbb{Z}X'$.

(a) Letting Y be an abelian group, exhibit an isomorphism of abelian groups

$$\hom_{\mathsf{Ab}}(\mathbb{Z}X, Y) \cong \hom_{\mathsf{Sets}}(X, Y).$$

On the left, we have the collection of abelian group homomorphisms from $\mathbb{Z}X$ to Y, while on the right, we have the collection of functions from X to the group Y.

(b) Let $f : X \to X'$ be any function, and $g : Y \to Y'$ a map of abelian groups (i.e., a group homomorphism). Show that your isomorphism from the previous problem fits into a commutative diagram

$$\begin{tikzcd}
\hom_{\mathsf{Ab}}(\mathbb{Z}X, Y') \arrow[r, "\cong"] & \hom_{\mathsf{Sets}}(X, Y') \\
\hom_{\mathsf{Ab}}(\mathbb{Z}X', Y) \arrow[r, "\cong"] \arrow[u, "(\mathbb{Z}f)^* g_*"] & \hom_{\mathsf{Sets}}(X', Y). \arrow[u, "f^* g_*"']
\end{tikzcd}$$

Remark I.12.1.2. The commutativity of this diagram is what makes the isomorphisms from the previous problem *natural*. (In math generally, the word natural is often used without precise meaning, but in category theory, the word "natural" is used for relations that are compatible with pre/post composition with functions. This is the sense in which we use the term "natural transformation" between two functors.)

Remark I.12.1.3. You have just exhibited the free-forget adjunction for abelian groups. Here, the functor sending a set X to the abelian group $\mathbb{Z}X$ is the left adjoint, and the functor sending an abelian group Y to its underlying set ("forgetting" the abelian group structure) is the right adjoint.

I.12.2. Smash product of sets. This exercise is meant to motivate the definition of smash product of spaces (by adopting the principle that spectra are like abelian groups, and that smash product of spectra is like tensor product).

(a) Exhibit an isomorphism $\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y \to \mathbb{Z}(X \times Y)$.

(Hint: Recall that the tensor product $A \otimes_{\mathbb{Z}} B$ is universal for bilinear maps out of $A \times B$ – that is, an abelian group homomorphism $q : A \otimes_{\mathbb{Z}} B \to C$ is the same thing as a function $f : A \times B \to C$ satisfying $f(a + a', b) = f(a, b) + f(a', b)$ and $f(a, b + b') = f(a, b) + f(a, b')$. If you can prove that abelian group homomorphisms $\mathbb{Z}(X \times Y) \to C$ are in bijection with such bilinear functions, and you can prove that your bijection is natural in the C variable, you have proven that $\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y$ and $\mathbb{Z}(X \times Y)$ are isomorphic; this is a consequence of the Yoneda Lemma.)

(You can alternatively ignore the previous hint if you have a favorite model for the tensor product, but the hint gives you a model-less way to think about how to prove two things are isomorphic.)

There is of course a forgetful functor from the category of groups to the category of sets, where any group is sent to its underlying set. However, group homomorphisms preserve particular elements (identity elements) so it is natural to try to understand a forgetful functor not to the category of sets, but to the category of pointed sets.

So suppose that (X, x_0) is a *pointed* set, meaning we have chosen an element $x_0 \in X$. A map of pointed sets is a function $f : X \to Y$ for which $f(x_0) = y_0$. Given a pointed set (X, x_0), one can define an abelian group

$$\text{Free}_*(X, x_0) := \mathbb{Z}X/\mathbb{Z}x_0.$$

That is, $\text{Free}_*(X, x_0)$ is obtained by quotienting the free Abelian group on X by the free subgroup generated by x_0.

(b) Verify that there is a free-forget adjunction between the category of groups and the category of pointed sets. (You can do this with as much detail as you want, but at the very least you should convince yourself that there is a natural bijection between the collection of abelian group homs from $\text{Free}_*(X, x_0)$ to an abelian group A, and the collection of basepoint-preserving functions from (X, x_0) to $(A, 0)$ where 0 is the identity element of A.)

(c) Let (X, x_0) and (Y, y_0) be pointed sets. Define (i.e., reverse-engineer) a new pointed set

$$(X, x_0) \boxtimes (Y, y_0)$$

so that you have a natural isomorphism

$$\text{Free}_*(X, x_0) \otimes_{\mathbb{Z}} \text{Free}_*(Y, y_0) \cong \text{Free}_*((X, x_0) \boxtimes (Y, y_0)).$$

(d) Compare your definition of $\boxtimes$ to the definition of smash product of pointed spaces.

I.13. Basic smash products of spaces

Let X and Y be two spaces with chosen basepoints x_0, y_0, respectively. Recall that the *smash product* of X with Y is denoted $X \wedge Y$, and is defined as the following quotient space:

$$X \wedge Y := (X \times Y)/\left(\{x_0\} \times Y \bigcup X \times \{y_0\}\right)$$

We take the image of $\{x_0\} \times Y \bigcup X \times \{y_0\}$ is taken as the basepoint of $X \wedge Y$, so that $X \wedge Y$ is not just a space, but a pointed space.

(a) Let X be a pointed space. Show that $S^0 \wedge X$ is homeomorphic to X as a pointed space.

(b) Let X be a pointed space and let $*$ be the one-point space. Compute $* \wedge X$.

(c) If you believe in the metaphor between smash of spectra and tensor of abelian groups, what does this say about $\Sigma^\infty(*) \wedge \Sigma^\infty X$? What abelian group should you interpret $\Sigma^\infty *$ to behave like?

(d) Let X be a pointed space and let ΣX be its reduced suspension. Show that $S^1 \wedge X \cong \Sigma X$. More generally, prove that $S^n \wedge X \cong \Sigma^n X$.

(e) Show that $S^n \wedge S^m \cong S^{n+m}$.

I.14. S^0 is a commutative ring

Let $\mathcal{C}$ be a category with a symmetric monoidal structure $\boxtimes$. For short we will write $\mathcal{C}^{\boxtimes}$ a sym Recall that part of the definition of symmetric monoidal structure guarantees the existence of an object 1 (called the unit) and the existence of natural isomorphisms

$$\sigma : A \boxtimes B \cong B \boxtimes A, \qquad A \boxtimes (B \boxtimes C) \cong (A \boxtimes B) \boxtimes C, \qquad \epsilon : 1 \boxtimes A \cong A,$$

and so forth. (We didn't write down all of the natural isomorphisms and their requirements.)

Then a *commutative algebra* or *commutative monoid* in $\mathcal{C}^{\boxtimes}$ is an object A, equipped with maps

$$m : A \boxtimes A \to A, \qquad u : 1 \to A$$

so that the following diagrams commute:

$$\begin{tikzcd} A \boxtimes A \arrow[r, "m"] \arrow[d, "\sigma"'] & A , \\ A \boxtimes A \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} 1 \boxtimes A \arrow[r, "\epsilon"] \arrow[d, "u \boxtimes \mathrm{id}_A"'] & A , \\ A \boxtimes A \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} A \boxtimes A \boxtimes A \arrow[r, "\mathrm{id}_A \boxtimes m"] \arrow[d, "m \boxtimes \mathrm{id}_A"'] & A \boxtimes A . \arrow[d, "m"] \\ A \boxtimes A \arrow[r, "m"] & A \end{tikzcd}$$

(a) If 1 is the unit of the symmetric monoidal structure, prove that 1 is a commutative algebra object.

(b) Write out some of the symmetric monoidal structure maps when $\mathcal{C} = \mathsf{Ab}$ is the category of abelian groups and $\boxtimes = \otimes_{\mathbb{Z}}$. (The unit object 1 is given by $\mathbb{Z}$.) Check that the commutative algebra structure you observed in the previous problem agrees with the usual multiplication of integers. (This is buried in the definition of the isomorphism $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$.)

(c) What is the commutative algebra structure on S^0 in the category of pointed topological spaces (with $\boxtimes$ given by the smash product)? It should be rather silly.

(d) Formulate what one should mean by a map of commutative algebras in $\mathcal{C}^{\boxtimes}$. Show that if A is a commutative algebra, then the map $u : 1 \to A$ is a map of commutative algebras.

Exercises about spectra

I.15. Homotopy groups of spectra

Let X be a spectrum. Recall this means X is the data of pointed spaces X_i for $i \geq 0$, along with homotopy equivalences $X_i \xrightarrow{\sim} \Omega X_{i+1}$ for all $i \geq 0$.

For any integer $i \in \mathbb{Z}$, we define the *ith homotopy group* of X to be

$$\pi_i(X) := \pi_{i-k}(X_k)$$

(a) Show that the definition of $\pi_i(X)$ is well-defined up to specified isomorphism of groups (implicitly, we are taking $k \geq i$ in the righthand side). Note this involves checking that the righthand side of the definition is independent of k (for $k \geq i$).

(b) If you know a definition of the sphere spectrum $\mathbb{S}$, show that $\pi_0(\mathbb{S}) \cong \mathbb{Z}$. Alternatively, show that $\pi_0(\Omega^k \Sigma^k S^0)$ is isomorphic to the integers.

(c) (*) It turns out $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2\mathbb{Z}$. Proving this usually requires the Freudenthal suspension theorem and a computation of $\pi_4(S^3)$. Exercise for the curious: Look up a proof that $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2\mathbb{Z}$.

I.16. Shifts and loops

It is clear from the definitions that there is a "shift" operation one can perform on spectra. That is, if the spectrum X is specified by the data $f_i : X_i \to \Omega X_{i+1}$ for $i \geq 0$, one can define two new spectra $X[1]$ and $X[-1]$ as follows:

$$X[1]_i := X_{i+1}, \qquad X[-1]_i := X_{i-1}$$

where we follow the convention that $X_{-1} = \Omega X_0$.

(a) Exhibit homotopy equivalences

$$\hom_{\mathsf{Spectra}}(X[-1], Y[-1]) \simeq \hom_{\mathsf{Spectra}}(X, Y) \simeq \hom_{\mathsf{Spectra}}(X[1], Y[1]).$$

Remark I.16.0.1. I've been dodgy about what we mean by composition of maps of spectra, but I hope this convinces you that the shift operation is an equivalence from Spectra to Spectra. You should think of this like the shift operation for chain complexes.

(b) Show that $[-1]$ is obtained by applying Ω to each space of a spectrum.

Remark I.16.0.2. For this reason, we often abuse notation and write

$$\Omega X := X[-1].$$

Indeed, writing ΩX instead of $X[-1]$ is often less confusing. The shift notation is notoriously confusing for people who switch between homological and cohomological conventions.

On the other hand, one must always be aware of whether we write ΩX for X a spectrum, or X a pointed space.

(c) For any spectrum X, convince yourself that ΩX is the homotopy pullback of the diagram

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & X. \end{tikzcd}$$

In other words, convince yourself that for any spectrum A, a homotopy from the zero map $A \to * \to X$ to itself is the same thing as a single map from A to ΩX. (If you have not seen this idea before for spaces, you should do this exercise when A and X are spaces.)

REMARK I.16.0.3. Here's a puzzle: The shift operation is obviously invertible for spectra. You can undo a shift of [1] by shifting [−1]. But Ω certainly was not invertible for spaces.

As mentioned already, spectra behave much more like chain complexes – shifts are invertible. This is an artifact of *stability* in the categorical sense, which we won't be able to discuss in these notes.

(d) Compute the homotopy groups of $X[1]$ and $X[-1]$ in terms of the homotopy groups of X.

I.17. Spectra are enriched over themselves

We won't prove the claim in the title of this exercise, but we will at least convince you that for any two spectra X and Y, the space $\hom_{\mathsf{Spectra}}(X, Y)$ is actually the 0th space of a spectrum.

(a) Let 0 denote the "zero spectrum," given by the spectrum whose ith space is a one-point space. (Any spectrum whose ith space is contractible is equivalent to the, and hence also called a, zero spectrum.) Show that there maps of spectra $X \to 0$ and $0 \to X$ – in fact, show that the space of maps $\hom_{\mathsf{Spectra}}(X, 0)$ and $\hom_{\mathsf{Spectra}}(0, X)$ are contractible.
(b) Conclude that given any two spectra X and Y, there exists a map $X \to Y$ factoring through 0. We will call this map (and any map homotopic to it) a zero map. The zero map renders the space of maps of spectra $\hom_{\mathsf{Spectra}}(X, Y)$ a pointed space.
(c) Using Exercise I.16, exhibit a homotopy equivalence

$$\hom_{\mathsf{Spectra}}(X, Y) \simeq \Omega \hom_{\mathsf{Spectra}}(X, Y[1]).$$

(d) Rinse and repeat to conclude that $\hom_{\mathsf{Spectra}}(X, Y)$ is the zeroth space of a spectrum.

Remark I.17.0.1. You can think of this as analogous to two classical facts: The set of abelian group homomorphisms from A to B is itself an abelian group, or the collection of chain maps between two chain complexes can be enriched to be part of a chain complex of morphisms. A pithy way to capture this is to say that the category of abelian groups, of chain complexes, and of spectra are enriched over themselves.

I.18. Some basic examples of spaces (not) arising from spectra

For each of the following spaces, prove whether the space does, or cannot, arise as the 0th space of a spectrum.

(1) A circle.
(2) A direct product of n circles.
(3) A direct product of a circle with $\mathbb{C}P^\infty$.
(4) The wedge of k circles where $k \geq 2$.
(5) A discrete space with 5 elements.
(6) A compact Riemann surface of genus $g \geq 2$.

I.19. Prespectra

Definition I.19.0.1. A *prespectrum* is the data of (i) a pointed space X_i for every $i \geq 0$, and (ii) a continuous map $\Sigma X_i \to X_{i+1}$ for every $i \geq 0$.

Example I.19.0.2. Let X_0 be any pointed space, and declare $X_i = \Sigma^i X_i$. By considering the identity maps $\Sigma X_i \to X_{i+1}$, we see that any pointed space X_0 defines a prespectrum.

A fancier example is to consider any vector bundle $E \to B$, and let X_i be the Thom space of the bundle $E \oplus \underline{\mathbb{R}^n}$ over B. (Here, $\underline{\mathbb{R}^n}$ is the trivial n-dimensional bundle over B.) This recovers the previous example when E is the trivial zero-dimensional vector bundle over $B = X_0$.

Remark I.19.0.3. The Thom space of a vector bundle E can be modeled as the one-point compactification of E if B is a nice space; alternatively one may choose a metric on E, and take the quotient of the unit disk bundle of E by the unit sphere bundle of E. It is an exercise to see that the Thom space $Th(E \oplus \underline{\mathbb{R}^n})$ is homeomorphic to $\Sigma^n Th(E)$. Indeed, that there were so many natural examples of such structures often led to many clunky definitions in the early history of spectra (like prespectra, and sometimes prespectra where all the maps $\Sigma X_i \to X_{i+1}$ were demanded to be homeomorphisms).

Given the proliferation of models, I should say that one's preferences often emerged not only out of which examples were natural, but also maps of spectra, and smash products of spectra, are.

I.19.1. From prespectra to spectra.

Remark I.19.1.1. Note that by adjunction, every prespectrum gives rise to a sequence of maps $X_i \to \Omega X_{i+1}$ for every $i \geq 0$, but these maps need not be homotopy equivalences. Regardless, by applying Ω^i to these maps, we obtain a sequence of maps

$$X_0 \to \ldots \to \Omega^i X_i \to \Omega^{i+1} X_{i+1} \to \ldots.$$

In other words, a pre-spectrum is precisely data that allows us to write down an "increasing sequence" – of a space X_0, into a group ΩX_1, into a more commutative group $\Omega^2 X_2$ (where this inclusion respects the group structure of X_1 because it is a map of loop spaces), into a more commutative group $\Omega^3 X_3$ (which respects the two-fold loop-space structure), and so forth. Naturally, the "union" of these spaces – that is, the colimit, taken appropriately – exhibits a space where we know how to multiply any two of its elements, and in a way as commutative as we like. This is what we do now:

Definition I.19.1.2 (The spectrum associated to a prespectrum). Fix a prespectrum X. Define a sequence of spaces $Y_i, i \geq 0$, by declaring

$$Y_i = \operatorname{colim}\Big(X_i \to \ldots \to \Omega^k X_{i+k} \to \Omega^{k+1} X_{i+k+1} \to \ldots\Big)$$

Here, the maps $\Omega^k X_{i+k} \to \Omega^{k+1} X_{i+k+1}$ are obtained by applying Ω^k to the map in Remark I.19.1.1.

Remark I.19.1.3. Technically, this colimit should be a homotopy colimit, so one could use a mapping telescope construction to define Y_i; alternatively, the homotopy colimit can be modeled as an honest union if we demand that every map $\Omega^i X_i \to \Omega^{i+1} X_{i+1}$ is a nice inclusion (more precisely, a cofibration). If you are not familiar with homotopy colimits, I encourage you to ignore these details, and I encourage you to think of every colimit of a sequence as above as simply an increasing union anyway.

(a) Prove that the collection $\{Y_i \to \Omega Y_{i+1}\}$ is indeed a spectrum.
    (Hint: By definition of the topology on an increasing union, and because S^1 is compact, a map from S^1 to an increasing union factors through a finite stage.)
(b) Maps of pre-spectra are slightly clunky to define, but an example of a map of prespectra is obtained by taking a collection of maps $g_i : X_i \to X_i'$ for which Σg_i and g_{i+1} are compatible in the obvious way. Show that such a collection gives rise to a map of spectra $Y \to Y'$.

Warning I.19.1.4 (Conflicting terms). A prespectrum is sometimes called a suspension spectrum in the literature. This is confusing because not every prespectrum is Σ^∞ of some space; and in our lectures, we reserve the term suspension spectrum for spectra that are equivalent to $\Sigma^\infty X$ for some pointed space X.

A prespectrum is also just called a spectrum in some older literature. This is also confusing and conflicts with the terminology in our course. Historically, the first examples of spectra really did look like prespectra: The sphere, Thom spectra, et cetera.

And, in some literature, what we call a spectrum is sometimes called an Omega-spectrum, or Ω-spectrum.

The proliferation of terminology is confusing, but it's not purely the fault of mathematicians. Spectra showed up in different contexts and looked slightly different in each context. See Section I.7.

I.20. Uniqueness of Eilenberg-MacLane spectra

Fix an abelian group A.

(a) Show that the space of homotopy autoequivalences of the Eilenberg-MacLane space $K(A,n)$ has π_0 given by $\mathrm{Aut}(A)$, the group of abelian group automorphisms of A. (Hint: There is an obvious map in one direction, given by π_n. In the other direction, you could take a CW model to create a map on the n-cells of $K(A,n)$ corresponding to the automorphism of A using a presentation of A; argue – using obstruction theory – that this extends to map from all of $K(A,n)$. Alternatively, you can use the well-known fact that $K(A,n)$ represents nth cohomology with coefficients in A.)

(b) Show that any two spectra with $\pi_0 \cong A$ as an abelian group and $\pi_i \cong 0$ for $i \neq 0$ are equivalent. (Hint: The constituent spaces X_i for such a spectrum are determined up to homotopy equivalence, so the only ambiguity is in the maps $f_i : X_i \xrightarrow{\sim} \Omega X_{i+1}$. Using the previous part, you can classify all such f_i up to homotopy; now write a map to your favorite Eilenberg-Maclane spectrum accordingly.)

LECTURE II

Smash product and free-forget

[1]

We saw in the last lecture the definition of spectra (Definition I.2.0.1). As advertised then, you should think of a spectrum like an abelian group, with the bonus that there is inherently topology involved. (For example, there is a notion of homotopy groups for spectra; and if you did the exercises, there are even notions of homotopy groups π_n with n negative.)

Roughly, spectra are "additive" objects, just as abelian groups are. But to work with categories or with rings, we need to know how to multiply. And to talk about multiplication or multilinear maps, the construction of tensor products is highly convenient. (A bilinear map $A \times B \to C$ is the same thing as a linear map $A \otimes_{\mathbb{Z}} B \to C$.)

In this lecture, we will give a philosophical way to think about the "tensor product for spectra." This tensor product is called the *smash product.* (Yes, this is the same name as the smash product for spaces; but we will denote the operation for spectra by $\otimes$, and for spaces by $\wedge$.)

My favorite way to motivate the smash product is to again think about the analogy to abelian groups, where we can understand the tensor product of abelian groups very conveniently by way of the free-forget adjunction. So we will begin there.

II.1. Free-forget for abelian groups

Recall that to any set X, one has the notion of the free abelian group $\mathbb{Z}X$ generated by X. This is an abelian group formed by taking the direct sum of $\mathbb{Z}$ "X many times." Naturally, any function $X \to Y$ gives rise to a map of abelian groups $\mathbb{Z}X \to \mathbb{Z}Y$.

Conversely, given any abelian group A, you can forget the group structure and just remember the underlying set of A.

A convenient way to encapsulate these operations is as follows: the "free" abelian group operation is a functor from the category of sets to the category of abelian groups

$$\mathrm{Free} : \mathsf{Sets} \longrightarrow \mathsf{Ab}$$

[1]**Note.** *This lecture is nearly identical to the fifth lecture I gave at the MSRI lecture series on spectra in Spring 2019 (during the Derived Algebraic Geometry program). In the present notes, I have added more background on the classical free-forget adjunctions.*

and the "forget" operation is a functor from the category of abelian groups to the category of sets

$$\mathsf{Sets} \longleftarrow \mathsf{Ab} : \mathrm{Forget}$$

Moreover, there is a natural bijection between the set of functions from X to A, and the set of (abelian) group homomorphisms from $\mathbb{Z}X$ to A:

$$\hom_{\mathsf{Ab}}(\mathbb{Z}X, A) \cong \hom_{\mathsf{Sets}}(X, A).$$

Using different notation, you might also write the above as:

$$\hom_{\mathsf{Ab}}(\mathrm{Free}(X), A) \cong \hom_{\mathsf{Sets}}(X, \mathrm{Forget}(A)). \tag{II.1.1}$$

(Concretely: To know a map from $\mathbb{Z}X$ to A is to know what the map does on a basis for $\mathbb{Z}X$, and the set X is a natural basis.) We say that the free functor is the *left adjoint* to the forget functor, and that forget is the right adjoint to the free functor.

$$\mathrm{Free} : \mathsf{Sets} \rightleftarrows \mathsf{Ab} : \mathrm{Forget}$$

This adjunction is called the free-forget adjunction.

Remark II.1.0.1 (Adjunctions, informally). Not everybody in the audience is familiar with category theory, so let me just say that "how many functions are there from U to V" is like[2] a"pairing" or a not-symmetric "inner product" on the collection of all objects in a category. Concretely, given two objects U and V, one can output a set (not a number) called $\hom(U, V)$.

If L and R were linear maps between inner product spaces, we would say R is an adjoint to L if $\langle Lu, v\rangle = \langle u, Rv\rangle$ for every u and v in the inner product spaces. Whether L is placed on the left, and R on the right, is of course important in the setting of infinite-dimensional vector spaces. Likewise, that Free is on the left, and Forget is on the right, is important in (II.1.1).

Just as adjoints are ubiquitous and important throughout linear algebra and functional analysis, adjunctions[3] are ubiquitous in most mathematics with useful structures.

But there is something slightly unnatural about the forgetful functor. The underlying set of an abelian group always has a distinguished element called the identity, and any group homomorphism respects this. There is accordingly an adjunction

$$\mathrm{Free}_* : \mathsf{Sets}_* \rightleftarrows \mathsf{Ab} : \mathrm{Forget}_*$$

where Sets_* denotes the category of *pointed* sets. (Objects are sets S together with a chosen element s_0, and morphisms are functions $S \to S'$ sending s_0

[2]Let me emphasize that "like" is important here; the following is an analogy, not an example. The analogy does, however, become more accurate if your category has some linear structure and one passes to its K theory.

[3]"Adjunction" is the category-theoretic noun for a pair of adjoint functors.

to s_0'.) What Free_* does is send a pointed set S to a free abelian group generated by S, modulo the copy of $\mathbb{Z}$ generated by s_0:

$$\mathrm{Free}_*(S) := \mathbb{Z}S/\mathbb{Z}\{s_0\}.$$

Informally, the basepoint s_0 "becomes" the unit of the abelian group. The free-forget adjunction from before in fact factors by the $(\mathrm{Free}_*, \mathrm{Forget}_*)$ adjunction preceded by an adjunction whose left adjoint sends a set S to a new set S_+; a basepoint is simply appended to the original set.

II.2. Free-forget and tensor products

Moreover, Free sends direct products of sets to tensor products (over $\mathbb{Z}$) of abelian groups; to be more precise, one can give Free the structure of a symmetric monoidal functor sending $\times$ to $\otimes_{\mathbb{Z}}$.

Remark II.2.0.1 (Symmetric monoidal stuff). Let me say briefly what a symmetric monoidal structure on a category, and symmetric monoidal structure on a functor, are. You can skip this whole remark if you are already familiar with these notions.

A *monoidal structure* on a category, roughly speaking, tells you how to take two objects and output a third. Just as a product is extra structure on a set, a monoidal structure is extra structure on a category. And just as a product on a set is a map $X \times X \to X$, a monoidal structure on a category contains data of a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Examples include $\otimes_{\mathbb{Z}}$ for abelian groups. Another monoidal strcuture is $\oplus$ for abelian groups. A key point is that a monoidal structure also comes equipped with natural isomorphisms $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$; this is a categorical version of "associativity up to homotopy" (note that the isomorphism is not an equality; this datum captures the idea of "associativity up to natural isomorphism"). One also usually supplies the data of a unit – that is, an object 1 along with natural isomorphisms $1 \otimes X \cong X \cong X \otimes 1$.[4]

Given a functor F from sets (with direct product) to abelian groups (with $\otimes_{\mathbb{Z}}$), a monoidal structure on F is a bunch of data, the most intuitive of which are equivalences $F(X \times Y) \cong F(X) \otimes_{\mathbb{Z}} F(Y)$ for every pair of objects X, Y. Note again this is a new collection of isomorphisms one has to specify on top of the functor F.[5] Whether composition is respected is not some property of F; it is extra structure one puts on F. A monoidal functor must also preserve the unit up to natural isomorphism.

The adjective *symmetric* in all this means one also provides data of natural swap isomorphisms. For example, to make a monoidal category a symmetric monoidal category, we provide data of natural isomorphisms $X \otimes$

[4] A subtle point is that if a unit exists, the unit and the natural isomorphisms are actually uniquely determined up to natural isomorphism, but it is often polite to provide the data of unit anyway.

[5] This is analogous to defining a map of "groups" in spaces $f : \Omega X \to \Omega Y$ not as just some continuous map, but a continuous map together with a homotopy $f \sim \Omega g$ to a map known to be a map of the form Ωg.

$Y \cong Y \otimes X$, along with compatibilities of this "commutative up to natural isomorphism" structure with the "associative up to natural isomorphism" structure.[6] Counterintuitively[7], once a functor is a monoidal functor between symmetric monoidal categories, the monoidal functor requires no additional structure to be considered symmetric.

You should think of a monoidal functor as like a map between monoids, while a symmetric monoidal functor is like a map between commutative monoids. A key fact we'll use over and over again is that a symmetric monoidal functor takes commutative objects in the domain category to commutative objects in the codomain category.

There is a natural symmetric monoidal structure on Sets_* that allows for Free_* to become symmetric monoidal, and that is the *smash product* on pointed sets. Given two pointed sets S and T, the smash product is defined to be

$$S \wedge T = (S \times T)/(S \times \{t_0\} \cup \{s_0\} \times T). \tag{II.2.1}$$

The reader can (and should) check easily that $\mathrm{Free}_*(S \wedge T) \cong \mathrm{Free}_*(S) \otimes_{\mathbb{Z}} \mathrm{Free}_*(T)$.

Remark II.2.0.2. The smash product for *spaces* can seem unmotivated at first, but the smash product of spaces has the exact same formula as (II.2.1). So the smash product of spaces obviously generalizes the smash product for sets, and you can see why somebody interested in a topological version of abelian groups might start considering it.

We begin in this chapter a study of the analogue for spectra. Rather than take a pointed set S and produce a free abelian group, we will take a pointed space X and produce a spectrum $\Sigma^\infty X$, which one can informally think of as the "free spectrum" generated by X. The analogue of the free-forget adjunction will be denoted

$$\Sigma^\infty : \mathsf{Spaces}_* \Longleftrightarrow \mathsf{Spectra} : \Omega^\infty$$

[6]A subtle point is that one must also demand that the composition of the swap isomorphisms $X \otimes Y \to Y \otimes X \to X \otimes Y$ be the identity map of $X \otimes Y$; otherwise one gets a *braided* monoidal category; the swap isomorphism encodes a nontrivial monodromy, if you like.

[7]This is an accident about the fact that categories only have *sets* of morphisms; for a category with *spaces* of morphisms, one should provide compatibilities between the swap maps; that is, a homotopy making the following diagram commute:

$$\begin{tikzcd} F(X \otimes Y) \arrow[r, "F(\simeq)"] \arrow[d, "\simeq"] & F(Y \times X) \arrow[d, "\simeq"] \\ F(X) \otimes F(Y) \arrow[r, "\simeq"] & F(Y) \otimes F(X). \end{tikzcd}$$

where the "forgetful" right adjoint will be notated by the popular Ω^∞ notation. The forgetful functor is quite concrete: It sends a spectrum to its 0th space.

Moreover, I assert here without proof that $\mathsf{Spectra}$ has a symmetric monoidal structure, which historically is also called "smash product;" we will denote this monoidal structure by any of the following symbols: $\otimes$ or $\otimes_{\mathbb{S}}$. Other works denote this by $\wedge$, which is the same symbol one uses for smash product of sets and spaces. For an overview of one way to set up the smash product, see Section II.11.

For us, and following the analogy with classical abelian groups, the key property will be that Σ^∞ can be promoted to a symmetric monoidal functor respecting smash product (i.e., sending $\wedge$ to $\otimes$). This is the content of Theorem II.6.0.1 below.

II.3. Free groups that are more and more commutative

Given a pointed space X, how would one construct a "free associative group" generated by X?

To start, let us assume X is a pointed *set*. We have a concrete set-theoretic construction of the free group generated by X (modulo the subgroup generated by the point). But we'd like something more clearly topological.

Let us consider the following topological construction: The reduced suspension ΣX of X is homotopy equivalent to a wedge of circles. A wedge of circles has no higher homotopy groups, and has π_1 isomorphic to the free group on $X \setminus \{x_0\}$. Thus its based loop space is homotopy equivalent to $\mathrm{Free}^{\text{Non-Ab}}(X)/\,\mathrm{Free}^{\text{Non-Ab}}(\{x_0\})$, and loop concatenation is compatible with the group structure on the free group.

This is where a picture ought to be to illustrate ΣX. If you want to provide Hiro with a good picture, please do. Hand-drawn pictures (see I.1.2.1) are welcome!

This construction – taking a pointed set X to the based loop space of its reduced suspension – actually works *all the time*. Put more precisely, consider the construction

$$\Omega\Sigma : \mathsf{Spaces}_* \to \mathbb{E}_1^{\mathrm{gp}}.$$

Here, the domain is the category of pointed spaces (morphisms are continuous maps respecting basepoints). The target, $\mathbb{E}_1^{\mathrm{gp}}$ (which stands for $\mathbb{E}_1$-groups), is the category of loop spaces – objects are loop spaces ΩY, and morphisms are maps of loop spaces, meaning maps $\Omega Y \to \Omega Z$ that arise as Ω of some continuous map $g : Y \to Z$ respecting basepoints.[8] The below

[8]We will encounter the term $\mathbb{E}_1$ again later; it is a synonym to the word A_∞. By definition, an $\mathbb{E}_1$-group is an A_∞-algebra whose π_0 is a group. It is not at all a priori obvious that an $\mathbb{E}_1$ group is the same thing as a based loop space. That an $\mathbb{E}_n$-group is the same thing as an n-fold loop space is May's recognition principle. Though this is a theorem, we treat it as an axiom for the expediency of exposition. (The recognition principle tells

proposition says that $\Omega\Sigma$ *is* the free functor, sending a pointed space to the free associative group generated by it.

PROPOSITION II.3.0.1. $\Omega\Sigma$ is left adjoint to the forgetful functor.

PROOF. We have that

$$\hom_{\mathbb{E}_1^{\mathrm{gp}}}(\Omega\Sigma X, \Omega Y) \simeq \hom_*(\Sigma X, Y) \simeq \hom_*(X, \Omega Y) = \hom_*(X, \mathrm{Forget}(\Omega Y))$$

and all the equivalences are natural. □

REMARK II.3.0.2. Let us give an informal motivation as to why the map $X \to \Omega\Sigma X$ is like a map from a pointed set to a free group on that pointed set (with prescribed identity). We will not be precise here. Any $x \in X$ determines a "longitudinal curve" in ΣX given by the image of the interval $[0,1] \times \{x\}$, so clearly any element of X defines an element of $\Omega\Sigma X$. On the other hand, each longitudinal curve can be run in reverse, so every element of X is now given some formal inverse. The topology of X is obviously respected, in the sense that if two points are nearby in X, then the associated longitudinal curves are nearby in a controlled sense. Finally, note that if X has multiple components, then the associated longitudinal loops have no relation between them; this is perhaps most easily seen when you take X to be a discrete set. So at least at the level of π_0, we do witness a (non-commutative) group emerging in $\Omega\Sigma X$ and the map $X \to \Omega\Sigma X$ plays the role of picking out the generators. Note that because the reduced suspension collapses the curve $[0,1] \times \{*\}$ to be constant, the basepoint does indeed play the role of a prescribed unit for this group.

More generally, $\Sigma^n X$ is a space which clearly has a longitudinal n-sphere arising naturally from every point of X (though it certainly contains more n-spheres), and $\Omega^n\Sigma^n X$ is a "$\mathbb{E}_n$-commutative" group by virtue of being an n-fold loop space.

PROPOSITION II.3.0.3. We have an adjunction

$$\Omega^n\Sigma^n : \mathsf{Spaces}_* \iff \mathbb{E}_n^{\mathrm{gp}} : U$$

between pointed spaces and $\mathbb{E}_n$-commutative groups (i.e., n-fold loop spaces). Here, U sends an n-fold loop space $\Omega^n Y$ to itself, as a pointed space.

By definition, objects of $\mathbb{E}_n^{\mathrm{gp}}$ are spaces of the form $\Omega^n Y$, and a map is a map of the form $\Omega^n g$ for some continuous $g : Y \to Z$ preserving basepoints.

PROOF. We again have

$$\hom_{\mathbb{E}_n^{\mathrm{gp}}}(\Omega^n\Sigma^n X, \Omega^n Y) \simeq \hom_*(\Sigma^n X, Y) \simeq \hom_*(X, \Omega^n Y).$$

□

us that this axiom is consistent, even if redundant, with the rest of mathematics.) The theorem was used implicitly in justifying the notion of "more and more commutative group" in Lecture One.

Take-away: The space $\Omega^n\Sigma^n X$ is the "free $\mathbb{E}_n$-commutative group" generated by X.

This take-away goes back to the work of Peter May[9].

II.4. Suspension spectra defined

Now, if you have a continuous function $f : S \to X$ between pointed spaces, you have an induced continuous function $\Sigma f : \Sigma S \to \Sigma X$ between their suspensions. So, the suspension functor defines maps

$$\hom_*(S^n, \Sigma^n X) \to \hom_*(S^{n+1}, \Sigma^{n+1} X).$$

(Here, $\hom_*$ is the space of continuous maps respecting basepoints.) These mapping spaces are, by definition, loop-spaces, so we can rewrite[10] the above as maps

$$\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X.$$

That is, each time we have a map of a sphere to (a suspension of) X, we can suspend that map to obtain a map from a higher-dimensional sphere to a higher-dimensional suspension of X.

Remark II.4.0.1. For any space X, one has a natural map $u_X : X \to \Omega\Sigma X$. One can think of this geometrically as "given a point of X, consider the longitudinal loop in ΣX passing through x." Categorically, this is the unit of the Σ-Ω adjunction. Then the map $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X$ can be realized as $\Omega^n u_{\Sigma^n X}$. In particular, it is a map of n-fold loop spaces.

So, Mother Nature has provided us with natural maps

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \Omega^3\Sigma^3 X \to \dots \tag{II.4.1}$$

and in light of the take-away from the previous section, we may interpret these maps as realizing more and more commutative groups generated by X. As curious humans, how can we *not* take the "limit" of this sequence?

That is, because we have a sequence of inclusions, we may as well take the increasing union; if you want a more point-set independent way of describing the result, we take the homotopy colimit (i.e., homotopy direct limit) of the above sequence.

Notation II.4.0.2 (QX). Let X be a pointed topological space. The colimit/increasing union of the sequence (II.4.1) is sometimes written

$$QX.$$

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[9] J. P. May, The Geometry of Iterated Loop Spaces, Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg 1972. A free and TeXed version is available on Peter's website.

[10] Be careful; many students for whatever reason are tempted to think $\Omega^n\Sigma^n = (\Omega\Sigma)^n$ but this is not true. See Exercise I.10

Construction II.4.0.3 (The suspension spectrum $\Sigma^\infty X$). Let X be a pointed topological space. Define a space

$$(\Sigma^\infty X)_0 = \mathrm{colim}_{n\geq 0}\, \Omega^n \Sigma^n X.$$

(This is the increasing union of the diagram pictured in (II.4.1).) More generally, we define

$$(\Sigma^\infty X)_i = \mathrm{colim}_{n\geq i}\, \Omega^{n-i} \Sigma^n X.$$

We have maps

$$\begin{aligned} \Omega((\Sigma^\infty X)_i) &\simeq \hom_*(S^1, (\Sigma^\infty X)_i) \\ &\simeq \hom_*(S^1, \mathrm{colim}_{n\geq i}\, \Omega^{n-i}\Sigma^n X) \\ &\simeq \mathrm{colim}_{n\geq i} \hom_*(S^1, \Omega^{n-i}\Sigma^n X) \\ &\simeq \mathrm{colim}_{n\geq i}\, \Omega^{n-(i-1)}\Sigma^n X \\ &\simeq (\Sigma^\infty X)_{i-1} \end{aligned} \tag{II.4.2}$$

thus exhibiting $\Sigma^\infty X$ as a spectrum.

Remark II.4.0.4. The equivalence (II.4.2) is a result of the circle being a compact topological space. Given an increasing sequence of spaces $A_i \subset A_{i+1} \subset \ldots$, if a compact space S maps to the union $\bigcup A_i$, compactness tells us that the map must factor through some finite stage of the infinite union. In particular, the mapping space from S to the union $\cup A_i$ is the union of the mapping spaces from S to each A_i.

This remark is heavily point-set inspired; but the same argument in the ∞-categorical setting yields the same result. For instance, any homotopy colimit of a sequence can be computed as an increasing union, simply by replacing each object and arrow in the sequence by homotopy equivalent data for which each arrow is a cofibration (a nice kind of injection). It's a theorem that when every arrow of a sequence is a nice inclusion, the sequential homotopy colimit can be computed as an honest sequential colimit (i.e., increasing union).

Definition II.4.0.5. We call $\Sigma^\infty X$ the *suspension spectrum* associated to the pointed space X.

Any spectrum homotopy equivalent to $\Sigma^\infty X$ for some X will also be called a suspension spectrum.

Remark II.4.0.6. A continuous map of pointed spaces induces a map of suspension spectra. We leave this verification to the reader, and for the reader to contemplate what it means for Σ^∞ to respect composition of functions in Exercise II.12.

Example II.4.0.7. Suspension spectra do not have negative homotopy groups. For example, in the example of the sphere spectrum, a negative homotopy group would have to come from $\pi_0\Omega^{n-i}S^n$ for some $i \geq 1$. But $\Omega^{n-i}S^n$ is the space of maps from the $(n-i)$-dimensional sphere into S^n, and all maps from lower-dimensional spheres to higher-dimensional spheres are homotopic to a constant map.

(Sketch of proof: You can homotope any map from a low-dimensional sphere to miss at least one point of S^n; that means the map factors through $\mathbb{R}^n \cong S^n \setminus *$, but this is contractible, so the map can further be contracted to be constant. This is the beginnings of cellular approximation, which says any k-dimensional complex maps in a way that can be homotoped to land in the k-skeleton of the target.)

If X is any topological space and $\Sigma^\infty X$ is its suspension spectrum, the same argument as before shows this when X is a CW complex, by cellular approximation. Because we only consider (in these lectures) spaces homotopy equivalent to a CW complex, and suspension preserves homotopy equivalences, the claim follows for such general X as well.

II.5. The sphere spectrum

Definition II.5.0.1. Let $X = S^0$ be the 0-sphere – the pointed space with exactly two points in it. Then we denote

$$\mathbb{S} := \Sigma^\infty S^0$$

and we call this suspension spectrum the *sphere spectrum*.

We will also often write

$$\mathbb{S}^n := \Sigma^\infty S^n.$$

Remark II.5.0.2. The pointed set with a single element has (reduced) free group given by the 0 group; and the pointed set with two elements has $\mathrm{Free}_*(S^0) = \mathbb{Z}$. Thus, whatever the sphere spectrum is, it should be thought of as the "free commutative group in spaces" generated by a single element. Its analogue over a classical commutative ring R would be the free rank 1 module over R.

Remark II.5.0.3. You might wonder, then, what QS^0 looks like. That is a phenomenal question.

The homotopy groups of QS^0 are called the stable homotopy groups of spheres.[11] It is a very wide-open problem to characterize the homotopy groups of this space in any meaningful fashion. What we have at present is the powerful machinery of *chromatic homotopy theory*, and the emerging *motivic homotopy theory*, to try and compute the homotopy groups of QS^0 one prime at a time. (It has been known since the work of Serre[12] that

[11] Why stable? First, it follows straight from the definitions that $\pi_i(QS^0) \cong \mathrm{colim}_{k\to\infty} \pi_{i+k}(S^k)$. By Freudenthal's Suspension Theorem this colimit/union stabilizes for k large enough. You can think of the term stable as referring to this stability, but now-a-days stable just means "what happens when you suspend a lot of times."

[12] Serre, Jean-Pierre (1951), "Homologie singulière des espaces fibrés. Applications", Annals of Mathematics, Second Series, 54 (3): 425–505, doi:10.2307/1969485, JSTOR 1969485, MR 0045386.

all stable homotopy groups, aside from $\pi_0 \cong \mathbb{Z}$, are finite groups.) Probably the current cutting edge has been achieved by Isaksen-Wang-Xu, and I highly recommend the introduction of their arXiv pre-print[13] to get a view of modern methods.

II.6. Smash product

We've constructed Σ^∞, which you should think of as the free functor taking a pointed space to the "free abelian group" (spectrum) generated by that space. Here is a theorem. You know almost enough to try and convince yourself of the first part of the theorem; that is left to you in II.13. The latter parts regarding the smash product, however, we will not attempt to prove.[14]

THEOREM II.6.0.1. There exists an adjunction of ∞-categories

$$\Sigma^\infty : \mathsf{Spaces}_* \rightleftarrows \mathsf{Spectra} : \Omega^\infty$$

where the left adjoint takes a pointed space to its suspension spectrum (Construction II.4.0.3) and the right adjoint takes a spectrum to its underlying space (the 0th space).

(a) Further, there exists a symmetric monoidal structure $\otimes$ on $\mathsf{Spectra}$ that allows Σ^∞ to be made symmetric monoidal with respect to smash product of spaces. In particular, one can supply natural equivalences

$$\Sigma^\infty(A \wedge B) \simeq (\Sigma^\infty A) \otimes (\Sigma^\infty B).$$

(b) Moreover, $\otimes$ is the *unique* symmetric monoidal structure that preserves colimits in each variable, and which renders Σ^∞ symmetric monoidal.

REMARK II.6.0.2. The above theorem characterizes the smash product, given that you know what spectra are. The ∞-category of spectra has a characterization as the universal stabilization of the ∞-category of spaces, but we won't be able to touch on that here.

REMARK II.6.0.3 (Some infinity-categorical ideas will be swept under the carpet). I used the term ∞-categories in the above theorem, and at present, this is by far the most natural way to articulate the theorem.

We have long been seeking the correct languages to describe systems containing rich homotopies. If you play with Fukaya categories, you know that A_∞-structures emerge inevitably, and that they are indispensible algebraic ingredients that help organize geometric phenomena. Higher homotopies serve analogous roles in modern homotopy theory. Because these things are as rich as they are important, the best methods we utilize take some time

[13] Daniel C. Isaksen, Guozhen Wang, Zhouli Xu. "More stable stems." arXiv:2001.04511.

[14] We give an outline of one proof in Section II.11.

to learn; and because these things take a long time to learn, people are invested in various methodologies, and this can lead to some incongruences in the community's philosophies.

One method for avoiding a lot of higher homotopies is the method of model categories, due to Quillen. While model categories have been hugely successful, and are often quite natural at organizing the homotopical structures in various settings, the biggest impediment to utilizing model category techniques is that model category structures often do not exist in examples of interest. Moreover, even in the settings where the necessary ingredients might exist, using only model categories to study homotopy theory is like defining manifolds as entities equipped with embeddings $X \subset \mathbb{R}^N$.[15] It is fine to do these embedding-dependent computations, but there are some ways to view particular manifolds that simply do not allow for efficient methodology if we must always employ an embedding into Euclidean space.

I bring all this up to say: At present, I do not know how to articulate the uniqueness in Theorem II.6.0.1 (b) using only the language of model categories, in any natural way.[16] It may even be that such a formulation of (b) cannot exist without using some ∞-categorical language.

By the way, one utility of ∞-categorical constructions made a minor appearance in Remark I.2.0.7; sometimes, it is best not to have to define composition, but to simply say what you would like of a homotopy-coherent diagram. Any demand that $[0,1]$ concatenated with itself must be equipped with a re-parametrization to fit into $[0,1]$ is unnatural from the ∞-categorical viewpoint; to expand the analogy further, one might view the Moore path space model as one "embedding of a manifold" for dealing with this issue.

Remark II.6.0.4 (Definitions of smash product). You are justified if you feel some frustration – what *is* $\otimes$? How is it defined?

You may see some models for $\otimes$ in later lectures by Cary; I also give some constructions in the exercises – see II.23 and II.24. But for now, you should feel a bit like a student who is learning the Eilenberg-Steenrod axioms, but who hasn't been given a model of any homology theories. Rest assured that there are models, but that you sometimes don't need these models when performing computations or concluding important facts. We will see how to conclude some important facts soon.

For the rest of this lecture, we will be content with item (b) of the theorem. It at least gives you a way to test whether a model is what it purports to be. If anybody tries to give you a formula for the smash product

[15] This was, for some time, the definition of manifold before the atlas definition came along.

[16] For example, even after fixing a model category for spaces, what if there are two natural symmetric monoidal model categories of spectra – both enjoying the expected Quillen adjunction with spaces – that do not admit a lax symmetric monoidal Quillen equivalence between them?

of spectra, but the model doesn't satisfy (b), run away and do not get in their van.

EXAMPLE II.6.0.5 (The sphere spectrum is a commutative ring). Let S^0 be the pointed topological space called the 0-dimensional sphere. Then S^0 is a commutative algebra in Spaces_* (see Exercise I.14). Because Σ^∞ is symmetric monoidal, it takes units to units; so the sphere spectrum $\Sigma^\infty S^0 = \mathbb{S}$ is the unit spectrum.

Exercise I.14 was about symmetric monoidal categories, and not symmetric monoidal ∞-categories. But the analogous statements hold in the ∞-categorical setting. In particular, the sphere spectrum is a commutative ring, and for any other commutative ring spectrum A, the unit map $u : \mathbb{S} \to A$ is a commutative ring map. In this sense, the sphere spectrum is the *initial* commutative ring.

REMARK II.6.0.6. It is a general fact of category theory that a right adjoint to a symmetric monoidal functor is lax symmetric monoidal. This holds in the ∞-categorical setting as well, so that Ω^∞ is lax symmetric monoidal. It's okay if you don't know what that means; but as a consequence, if Y is a spectrum with some multiplicative structure, then the space $\Omega^\infty Y$ will also have that multiplicative structure.

In light of Example II.6.0.5, this means that $QS^0 = \Omega^\infty \mathbb{S}$ does not just have a homotopical abelian group structure (given by the colimit of $\Omega^n S^n$) but also has a multiplicative structure that is linear over this abelian group structure. See Exercise II.21.

II.7. Freudenthal suspension theorem (not covered in lecture)

This section is for computational purposes; it is not necessary to setting up any of the theory, but most of us need to compute something at some point in our research.

Big colimits, even if they're nice sequential ones, can look a bit intimidating at first glance. For instance, even if you wanted to compute π_0 of the 0th space of a suspension spectrum, how would you do it?

Though this is historically backward, the *Freudenthal suspension theorem* allows you to compute these homotopy groups at a finite stage of the colimit. (Historically, the Freudenthal suspension theorem is one of the things that encouraged people to pass to sequences of suspensions to study stable homotopy groups.) The theorem says that if X has no topology between dimensions 0 and n, then up to *twice* the dimension of n, homotopy groups are unchanged by taking free groups.

THEOREM II.7.0.1 (Freudenthal suspension theorem). Let X be n-connected, meaning that $\pi_i = 0$ for all $i \leq n$. Then the natural map

$$\pi_k(X) \to \pi_k(\Omega\Sigma X)$$

is an isomorphism for $k \leq 2n$. It's in fact a surjection for $k = 2n + 1$.

We will not give a proof here.

REMARK II.7.0.2. The "natural map" of the theorem is the one induced by the suspension functor:

$$\mathrm{hom}_*(S^k, X) \to \mathrm{hom}_*(\Sigma S^k, \Sigma X) \cong \mathrm{hom}_*(S^k, \Omega\Sigma X).$$

Now just apply π_0 on both sides. The reason we present the theorem in the above way is to simplify the indexing of the k and n indices.

EXAMPLE II.7.0.3. Let X be the 2-sphere, which is 1-connected (i.e., $n = 1$). Then the map

$$\pi_3(S^2) \to \pi_3(\Omega S^3) \cong \pi_4(S^3)$$

is a surjection and

$$\pi_{n+1}(S^n) \to \pi_{n+2}(S^{n+1})$$

is an isomorphism for all $n \geq 3$. So the homotopy group π_1 of the sphere spectrum is computed by identifying $\pi_4(S^3)$. It turns out that this group is $\mathbb{Z}/2\mathbb{Z}$, and its non-trivial generator is hit by the Hopf element of $\pi_3(S^2)$; this is the attaching map that creates $\mathbb{C}P^2$ out of $\mathbb{C}P^1$.

EXAMPLE II.7.0.4 (Homotopy groups of suspension spectra). Let us compute some of the homotopy groups of suspension spectra. We already saw the negative homotopy groups are zero by a cellular approximation argument.

We have that

$$\pi_i(\Sigma^\infty X) \cong \pi_i(\mathrm{colim}_{n\to\infty} \Omega^n\Sigma^n X) \cong \mathrm{colim}_{n\to\infty} \pi_{n+i}(\Sigma^n X).$$

By the Freudenthal suspension theorem, this stabilizes for large values of n; these are called the *stable homotopy groups of* X.

(Note that ΣX is always connected, $\Sigma^2 X$ is always 1-connected, and $\Sigma^n X$ is always $(n-1)$-connected.)

II.8. Why ∞-categories? (Not covered in lecture)

There are many compelling reasons to talk about ∞-categories; this week, the most compelling reason not to delve into them is (unfortunately) time.

But let me just give an indication of the difficulties one would encounter if one tries to realize everything covered so far in the world of categories, and not of ∞-categories.

In what follows, recall that for a pointed topological space X, we have

$$QX := \mathrm{colim}_n \Omega^n\Sigma^n X$$

to be the free abelian group generated by X. We know that it models the 0th space of the suspension spectrum of X. There is the following no-go theorem of Lewis from 1991:

Theorem II.8.0.1 (Lewis[17]). Suppose $\mathcal{T}$ is the category of pointed spaces, and $\mathcal{S}$ is some other category. Consider the following axioms:

(1) $\mathcal{S}$ has a symmetric monoidal structure $\otimes$.
(2) There exists an adjunction $\Sigma^\infty : \mathcal{T} \iff \mathcal{S} : \Omega^\infty$.
(3) The unit for $\otimes$ is $\Sigma^\infty S^0$.
(4) Either Ω^∞ is lax monoidal with respect to $\otimes$ and $\wedge$, or Σ^∞ is op-lax monoidal.
(5) There is a natural weak equivalence $\Omega^\infty\Sigma^\infty X \to QX$ factoring the inclusion $X \to QX$ through the unit of the adjunction.

There does not exist a symmetric monoidal category $(\mathcal{S}, \otimes)$ satisfying the above axioms.

Lewis knew full well the state of the field at the time of his article. Here is the abstract from the paper:

> The construction of the smash product of two spectra is one of the most unsatisfactory aspects of every available treatment of the stable category. Increased interest in enriched ring and module spectra has made the misbehavior of smash products a source of growing frustration. This paper conveys the unhappy message that this frustration is unavoidable. Five simple, obviously desirable axioms for a good category of spectra with a well-behaved smash product are listed. Then it is shown that no category can satisfy all five of these minimal axioms.

In contrast, by replacing the word category with ∞-category, we can exhibit $\mathsf{Spectra}$ and Spaces_* satisfying all the above axioms (in fact, Σ^∞ is symmetric monoidal).

Of course, before the advent of ∞-categories, many topologists got around the above no-go theorem. For example, one can relax unitality (expressed via isomorphisms) to be expressed via weak equivalences.

II.9. The Pontrjagin-Thom theorem (not covered in lecture)

These notes hint at at least two reasons that the sphere spectrum $\mathbb{S}$ ought to have a ring structure. First, $\mathbb{S}$ is the unit of a symmetric monoidal structure (which makes $\mathbb{S}$ not only a ring but a commutative ring) – this is motivated in Exercise I.14. Second, $\mathbb{S}$ is also realized as an endomorphism algebra (this shows at least that $\mathbb{S}$ is an associative ring) – see Exercise II.21. Let us give a third reason here, whose origins are geometric.

Construction II.9.0.1 (The space of framed cobordisms). We define a space P combinatorially – i.e., by defining its collection P_k of k-simplices. For every integer $k \geq 0$, an element of P_k is roughly the data of

[17] L. Gaunce Lewis, "Is there a convenient category of spectra?", Journal of Pure and Applied Algebra, Volume 73, Issue 3, 1991, Pages 233-246, ISSN 0022-4049. `https://doi.org/10.1016/0022-4049(91)90030-6`.

(i) A compact subset $X \subset \mathbb{R}^\infty \times \Delta^k$ which is a smooth manifold, possibly with corners, whose projection to Δ^k respects codimension. (Note that the empty manifold is a manifold of every codimension, so X can be a closed manifold so long as the embedding has no fibers above $\partial\Delta^k$.)
(ii) A trivialization (i.e., a framing) of the normal bundle of X.

We call P the space of framed cobordisms.

WARNING II.9.0.2. The notation P is not standard. There is, as far as I know, not a standard notation for the space of framed cobordisms.

EXAMPLE II.9.0.3. The vertices of P are given by normally framed, compact, 0-dimensional manifolds embedded in $\mathbb{R}^\infty$. You can think of a vertex, hence, as a collection of points, each point assigned a plus or minus sign. A 1-simplex between two such vertices is a framed cobordism between these points. In particular, we see that $\pi_0 P \cong \mathbb{Z}$, given by the signed count of points.

More generally, the homotopy groups of P are the framed cobordism groups.

THEOREM II.9.0.4 (Classical Pontrjagin-Thom Theorem). For all $i \geq 0$, there exists an isomorphism

$$\pi_i(P) \cong \pi_i(QS^0).$$

In other words, the stable homotopy groups of spheres compute the framed cobordism groups of manifolds.

Now, because $\mathbb{S}$ is a ring spectrum, its homotopy groups (hence the homotopy groups of QS^0) form a graded ring – see Exercise II.22. Is there an additive, and a multiplicative, structure on the space of framed cobordisms?

Yes. Given two closed framed i-manifolds, their disjoint union is also a closed framed i-manifold. This gives the addition on $\pi_i(P)$. And the direct product of two closed framed manifolds of dimensions j and k is again a closed framed manifold. This gives the map $\pi_i(P) \times \pi_j(P) \to \pi_{i+j}(P)$.

But, of course, we would like to realize these operations at the π_* level as "space-level" operations – that is, write a map $P \times P \to P$ encoding the disjoint-union-of-manifolds operation, and a map $P \times P \to P$ encoding the direct-product-of-manifolds operation. Indeed, a commonly accepted[18] form of the Pontrjagin-Thom theorem states that one can create such a model for operations on P rendering it a ring, and that the Pontrjagin-Thom isomorphism on π_* in fact arises from an equivalence of rings $P \to QS^0$.

In other words, the ring structure of the sphere – where addition arises from the pinching of i-dimensional spheres, and where multiplication arises from the fact that $S^i \wedge S^k \cong S^{i+k}$ – is equivalent to the ring structure of the space of framed cobordisms. I don't know which incarnation is more fundamental and basic, but this is a beautiful fact that resonates throughout modern topology. This fact, and its descendants, allow us to make progress

[18] But I do not know where, or if, it is written.

in the classification of smooth manifolds by performing computations in spectra.

REMARK II.9.0.5. We take the Pontrjagin-Thom theorem to illustrate two philosophical points. First, it makes us think that both stable homotopy theory and cobordism theory are important. (When two natural, nuanced, and disparate machines recover each other, it seems that mother nature is pointing us in the right direction.)

Second, the theorem illustrates a phenomenon wherein algebraic structures of geometric origins rarely want "strict" or "equality-based" characterizations – there is no one way to embed a disjoint union of manifolds. One can appreciate why, in setting up spectra, one would like a framework flexible enough to accommodate geometry as giant and convenient as the space of embeddings of manifolds into $\mathbb{R}^\infty$.

II.10. Generalized cohomology theories and Brown representability (not covered in lecture)

Whatever one meant by spectra, it was known early on that spectra give rise to "generalized cohomology theories."

Let us motivate this term. First, when A is an abelian group, and HA is the associated spectrum (i.e., the Eilenberg-MacLane spectrum), we can take a space X and perform the following sequence of operations

$$X \mapsto \Sigma^\infty X \mapsto \hom_{\mathsf{Spectra}}(\Sigma^\infty X, HA)$$

and compute the homotopy groups of this mapping space. In fact, this mapping spaces is itself the 0th space of a natural spectrum – see Exercise I.17 – so one can compute its negative homotopy groups as well. It turns out that π_{-i} recovers precisely the reduced cohomology $\widetilde{H}^i(X;A)$ – see Exercise II.26.

More generally, for any spectrum Y, the assignment

$$X \mapsto \pi_* \hom_{\mathsf{Spectra}}(\Sigma^\infty X, Y)$$

satisfies a generalized version of the Eilenberg-Steenrod axioms for cohomology. In what probably seemed like a deep miracle at the time, the *Brown representability theorem* discovered that *any* graded-abelian-group invariant of spaces satisfying such axioms *must* arise form a spectrum Y via the above construction.

For this reason, it was (and still is, in some circles) common to think of spectra as generalized cohomology theories.

However, spectra are far richer than the cohomology theories to which they give rise. This is not obvious at the level of objects, in that two equivalent cohomology theories do give rise to equivalent spectra. What matters is the morphisms. There exist morphisms between spectra that give rise to the zero map in the respective cohomology theories, but that are not (homotopic to) a zero map between spectra.

II.11. How do we prove the existence of the smash product? (not covered in lecture)

The hardest parts of Theorem II.6.0.1 are that there is a symmetric monoidal structure on $\mathsf{Spectra}$, and that is it the unique one satisfying certain properties. As I've hinted many times in these notes, it seemed an eternal struggle (lasting decades) in homotopy theory to pin down such a construction. The struggle led to many people developing amazing understanding of things like finite sets and bijections, the geometry of how Euclidean spaces and their isometries behave as we increase dimension, configurations of points, et cetera, et cetera, et cetera. The problem of course is that when you develop different models, our techniques become different from one community to the next, and our research community can fracture. Imagine being a student being raised on one diet of spectra, only to find as you develop that different communities digest different kinds of spectra.

Remark II.11.0.1. Flipping the abundance of models on its head, what we realize is that there ought to be one common principle that explains why each of these models are supposed to be morally correct. To create a mathematical framework in which this principle is a theorem would be the hare-brained approach (see Adams's quote in Section I.7; "hare-brained" is not meant to be diminutive).

Amazingly, the technique that I learned (from Lurie's writings) to prove Theorem II.6.0.1 involves no knowledge of the intricate geometric studies I just listed. It is proven by moving in a completely different academic direction – by utilizing higher category theory.

There's a disgustingly slick trick that allows us to prove the theorem. For expediency I will not define all the terms. The main reference is Section 4.8.2 of Lurie's Higher Algebra, freely available from Lurie's web page. You can also take a look at David Gepner's chapter in the Handbook of Algebraic Topology[19].

Lemma II.11.0.2. In any monoidal ∞-category $\mathcal{C}^{\otimes}$ (the superscript $\otimes$ is to pin down the notation for the monoidal product), the following data are equivalent:

(a) A unital associative algebra[20] R for which the multiplication map $R \otimes R \to R$ is an equivalence.
(b) A localization[21] functor $L : \mathcal{C} \to \mathcal{C}$ of the form $X \mapsto X \otimes R$.

---

[19] David Gepner, "An Introduction to Higher Categorical Algebra," arXiv:1907.02904.

[20] We have not defined this explicitly yet, but you can imagine there is a way to codify when a product is associative up to higher and higher homotopies, similar to our approach in Lecture I.

[21] Roughly, this means that for any object B in the image, we have a natural equivalence $\hom_{\mathcal{C}}(A, B) \simeq \hom_{L\mathcal{C}}(LA, B)$.

Moreover, when $\mathcal{C}^{\otimes}$ is symmetric monoidal, the image $L\mathcal{C}$ inherits a symmetric monoidal structure for which L is a symmetric monoidal functor. In particular, R inherits a canonical commutative algebra structure.

REMARK II.11.0.3. If you are familiar enough with categories to contemplate the above lemma, I encourage you to do so. After working out a few examples, it becomes compellingly true, and it's the kind of thing a lot of experts just "know" even without proof. For a proof and discussion, see 4.8.2 of Lurie's Higher Algebra. Another resource is a paper of Lazarev-Sylvan-Tanaka[22], where we unknowingly reproduced things that were already contained in Higher Algebra[23].

EXAMPLE II.11.0.4. The prototypical example is to take $\mathcal{C} = \mathsf{Ab}$ to be the category of abelian groups, with $\otimes_{\mathbb{Z}}$, and choose $R = \mathbb{Z}[1/p]$ for some prime p. This example also gives some idea of the origins of the word "localization," as tensoring with $\mathbb{Z}[1/p]$ is like restricting a sheaf on $\mathrm{Spec}(\mathbb{Z})$ to an open subset of $\mathrm{Spec}(\mathbb{Z})$.

The example in which we apply the lemma is as follows: We let $\mathcal{P}r^L$ be the symmetric monoidal category of presentable ∞-categories, and whose functors are those that preserve all colimits[24]. Informally, this means that an element A of this category is an ∞-category admitting all limits and colimits, and is generated under colimits by a small collection of objects. We define the tensor product on $\mathcal{P}r^L$ to be the universal one which preserves all small colimits in each variable.[25] The unit of $\mathcal{P}r^L$ is the ∞-category Spaces of spaces.

To prove Theorem II.6.0.1, it thus suffices to show that the ∞-category of spectra $\mathsf{Spectra}$ is an idempotent in $\mathcal{P}r^L$, with the unit map $\mathsf{Spaces} \to \mathsf{Spectra}$ given by the functor $X \mapsto \Sigma^\infty(X_+)$.

The proof of this fact relies on the equivalence in the lemma above, by showing that there is a universal way to turn a presentable ∞-category into a stable one. This universal way, called the stabilization process, is easily proven to be a localization of $\mathcal{P}r^L$. Roughly, and in only slightly more concrete terms, this comes down to the fact that a colimit-preserving functor

[22]Localization and flexibilization in symplectic geometry, with Oleg Lazarev and Zachary Sylvan. arXiv:2109.06069

[23]It is very common in the field to discover a fact about ∞-categories that was already contained somewhere in Lurie's works.

[24]The Pr is for "presentable." The L is for "left adjoints," because colimit-preserving functors between presentable ∞-categories are always left adjoints.

[25]The construction of this $\otimes$ is formal, but is only formal if one uses facts about ∞-categories that are most easily proven using the combinatorial models due to Joyal and Lurie. Indeed, these "formal" proofs rely on the two ideas of marked simplicial sets and (co)Cartesian/categorical fibrations of simplical sets – the absence of (easily articulable analogues of) these concepts in other models of higher categories is one huge reason that ∞-categories are, at present, the most efficient model for proving formal results in higher category theory.

from A to a stable B is the same thing as a map from the stabilization of A to B.

So one abstractly finds a symmetric monoidal structure on $\mathsf{Spectra}$. How do we compute anything about it? We indicate how in the exercises; see in particular Exercise II.24.

Exercises

II.12. Σ^∞ on morphisms

Fix a continuous map of pointed spaces $f : X \to Y$.

(a) Convince yourself that f induces a map of spectra $\Sigma^\infty X \to \Sigma^\infty Y$.

(b) What can you say about how Σ^∞ respects "composition?" (It may help to read Remark I.2.0.7.)

II.13. The $(\Sigma^\infty,\Omega^\infty)$ adjunction

(This exercise assumes some familiarity with category theory tools; namely, using Kan extensions to compute limits step by step. Even if you don't know much about these techniques, it may be a good opportunity to look under the hood of the vehicles homotopy theorists tend to drive.)

Prove the following theorem:

Theorem II.13.0.1. For any pointed space A and any spectrum Y, there is an equivalence of mapping spaces

$$\hom_{\mathsf{Spectra}}(\Sigma^\infty A, Y) \simeq \hom_{\mathsf{Spaces}_*}(A, Y_0).$$

Hint of a proof. Unless equipped with a subscript, hom always means $\hom_{\mathsf{Spaces}_*}$.

Consider the staircase-shaped diagram

$$\begin{tikzcd}
\hom(A, Y_0) & & & \\
\hom(\Omega\Sigma A, Y_0) \arrow[u] & \hom(\Sigma A, Y_1) \arrow[l] & & \\
\hom(\Omega^2\Sigma^2 A, Y_0) \arrow[u] & \hom(\Omega\Sigma^2 A, Y_1) \arrow[l] \arrow[u] & \hom(\Sigma^2 A, Y_2) \arrow[l] & \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}$$

The limit of the diagram can be computed as the right Kan extension along the collapse of the diagram to a point. One can factor this collapse in two ways: First by collapsing the columns of the diagram, or by first collapsing the rows.

Collapsing the columns, one obtains a sequential inverse limit diagram whose entries are computed by the computing the limits of each column. The limit of the ith column is by definition

$$\hom(Q\Sigma^i A, Y_i) \simeq \hom(\operatorname{colim} \Omega^n \Sigma^{n+i} A, Y_i) \simeq \lim \hom(\Omega^n \Sigma^{n+i} A, Y_i)$$

so the limit of the staircase diagram above is computed as

$$\lim \left(\hom(QA, Y_0) \xleftarrow{\Omega} \hom(Q\Sigma A, Y_1) \xleftarrow{\Omega} \ldots \right).$$

Here, the notation Ω is shorthand for the composite

$$\hom(Q\Sigma^i A, Y_i) \simeq \hom(\Omega Q \Sigma^{i+1} A, \Omega Y_{i+1}) \xleftarrow{\Omega} \hom(Q\Sigma^{i+1} A, Y_{i+1}).$$

This sequential limit is, by definition, homotopy equivalent to $\hom_{\mathsf{Spectra}}(\Sigma^\infty A, Y)$.

Now let us compute the limit of the staircase diagram by first collapsing the rows. Each row has an initial vertex given by $\hom(\Sigma^i A, Y_i)$; by definition of limit, these initial vertices are the limits of each row. By tracing through the definitions, one recognizes the maps between these row-wise limits as realizing the (Σ, Ω)-adjunction (or the free-forget adjunction for $\mathbb{E}_n$-commutative groups), so the limit of the staircase diagram is computed as the limit of the tower of equivalences

$$\begin{tikzcd} \hom(A, Y_0) \\ \hom(\Sigma A, Y_1) \arrow[u, "\sim"] \\ \vdots \arrow[u, "\sim"] \end{tikzcd}$$

hence is equivalent to $\hom(A, Y_0)$.

□

II.14. Homotopy groups of spectra are detected by the sphere spectrum.

(1) Using the free-forget adjunction for spectra, prove the positive homotopy groups of the spectrum X are precisely the homotopy groups of the mapping space of maps from the sphere spectrum to X. (This is analogous to how the cohomology groups of a chain complex A are the cohomology groups of the hom cochain complex $\hom(\mathbb{Z}, A)$.)

Remark II.14.0.1. In fact, we saw evidence in Exercise I.17 that $\mathsf{Spectra}$ is enriched over itself; so the mapping spectrum from $\mathbb{S}$ actually recovers all homotopy groups of X.

(2) For all $n \geq 0$, there is a pinch map $S^n \to S^n \vee S^n$ (Exercise I.11). Because the wedge sum $\vee$ is the coproduct of pointed spaces and

Σ^∞ preserves colimits, there is an induced map $\mathbb{S}^n \to \mathbb{S}^n \oplus \mathbb{S}^n$ of spectra; hence there is an induced map

$$\hom_{\mathsf{Spectra}}(\mathbb{S}^n, X) \times \hom_{\mathsf{Spectra}}(\mathbb{S}^n, X) \to \hom_{\mathsf{Spectra}}(\mathbb{S}^n, X).$$

Taking π_0 of these mapping spaces, we thus have a map $\pi_n(X) \times \pi_n(X) \to \pi_n(X)$. Show that this agrees with the usual addition on $\pi_n(X) = \pi_n(X_0)$.

II.15. Shifts and suspensions

Given that Σ^∞ is a left adjoint, it preserves colimits.[26]

(a) Recalling the definition of the (reduced) suspension of a pointed space, show it follows straight from the definitions that

$$(\Sigma^\infty(\Sigma X))_i \simeq (\Sigma^\infty X)_{i+1}.$$

(b) Let A and B be pointed topological spaces. Convince yourself that a pointed continuous map from ΣA to B is the same thing as a homotopy from the constant map $A \to B$ to itself (through maps of pointed topological spaces). In other words, the homotopy pushout of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

is given by the homotopy commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma A \end{tikzcd}$$

(where the "square" itself encodes a homotopy from the constant map $A \to \Sigma A$ to itself).

(c) Given a spectrum $X = (X_i, f_i)$ one can define two different shifts of the spectrum – by declaring $(X[1])_i = X_{i-1}$ or $(X[-1])_i = X_{i+1}$. By using the fact that Σ^∞ preserves colimits – and hence homotopy pushouts – conclude that when X is a suspension spectrum, one of these shifts is in fact a homotopy pushout of X along two zero maps.

(d) Conclude that, for spectra, Σ and Ω are mutually inverse. (See Exercise I.16.)

(e) By example, demonstrate that Σ and Ω are not mutually inverse operations on topological spaces.

[26]More accurately, Σ^∞ preserves all homotopy colimits; it only preserves colimits for very particular models of spaces, spectra, and of Σ^∞. Any reasonable model of Σ^∞, on the other hand, preserves all homotopy colimits. If this distinction doesn't mean much to you, you can ignore it.

Remark II.15.0.1. Indeed, one way to think of the theory of spectra is as the theory obtained from spaces by universally inverting Σ (or Ω). The word theory is a bit vague; one can make this precise using the language of ∞-categories.

II.16. Spectra arise from shifts of suspension spectra

Given a spectrum X, let $\Sigma^{-n}X$ denote the spectrum whose ith space is given by X_{i-n}. (One could likewise notate this spectrum as $\Omega^n X$, or as $X[-n]$.)

(a) Show that any spectrum X, with nth space given by X_n, admits a natural map

$$\left(\operatorname{colim}_{n\to\infty} \Sigma^{-n}\Sigma^{\infty}X_n\right) \to X. \tag{II.16.1}$$

(Hint I: There is a natural map $\Sigma^{\infty}\Omega A \to \Omega\Sigma^{\infty}A$ for any pointed space A. Here, we interpret Ω of a spectrum as a homotopy pullback along the zero map, or as a shift of A. See Exercise II.15(c).)

(Hint II: The shift functor is an autoequivalence of $\mathsf{Spectra}$; hence it preserves all limits. Thus, the composition $\Omega^{\infty-n} := \Omega^{\infty}\circ\Omega^{-n}$ preserves all limits, so there is a left adjoint from spaces to spectra that one might denote by $\Sigma^{\infty-n}$. Identify this left adjoint with $\Sigma^{-n}\Sigma^{\infty}$. Use this to conclude that there are natural maps $\Sigma^{-n}\Sigma^{\infty}X_n \to X$. You'll want to show that these two hints are compatible to exhibit the map (II.16.1).)

(b) (*) Prove that the above map is an equivalence. (Hint: Using the fact colimits of domain objects turn into limits of mapping spaces, and using adjunctions deftly, exhibit a map from $\hom_{\mathsf{Spectra}}(X,Y)$ into a limit of mapping spaces $\lim \hom_*(X_n,Y_n)$ – tracing through your use of adjunctions, interpret this limit as picking out exactly those maps $X_n \to Y_n$ that respect the delooping maps $X_i \simeq \Omega X_{i+1}$ and $Y_i \simeq \Omega Y_{i+1}$. Conclude that both sides of (II.16.1) corepresent the same functor.) Note that this proves that every spectrum is a colimit of shifted suspension spectra.

II.17. Long exact sequences for homotopy groups

(a) Suppose that

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r, "b_0"] & B \end{tikzcd}$$

is a fiber sequence of spaces. This means p is a fibration (if you're not familiar, you can pretend this means p is a fiber bundle) and F is the fiber of p above b_0. Look up (don't spend your time proving) the fact that there is a long exact sequence of homotopy groups

$$\ldots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \ldots.$$

Just try to understand what this means; you will use it soon.

(b) Now suppose that the above diagram is more generally a homotopy pullback diagram.[27] (It is a fact that fiber sequences are homotopy pullbacks.) Convince yourself that the top square, and the vertical rectangle, in

$$\begin{tikzcd} \Omega B \arrow[r] \arrow[d] & * \arrow[d] \\ F \arrow[r, "i"] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r, "b_0"] & B \end{tikzcd}$$

are both homotopy pullback diagrams. (You will want to supply the homotopies for both diagrams.)

(c) Using the fact that a homotopy from the constant map $S^n \to B$ to itself defines a map $S^{n+1} \to B$, prove there is a long exact sequence of homotopy groups

$$\ldots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \ldots.$$

This proof using the homotopy pullback property is different from, say, the proof you find in Hatcher. This in particular proves the long exact sequence from part (a).

(d) (*) More generally, if one has a homotopy pullback square of spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ B' \arrow[r] & C \end{tikzcd}$$

show one has a long exact sequence of homotopy groups

$$\ldots \to \pi_k(A) \to \pi_k(B) \oplus \pi_k(B') \to \pi_k(C) \to \pi_{k-1}(A) \to \ldots.$$

Remark II.17.0.1. Philosophically, while homology is well-behaved when gluing things together (see, e.g., Mayer-Vietoris) homotopy groups are well-behaved for homotopy-fibering spaces together.

(e) By definition, a *homotopy pull-back diagram* in an ∞-category $\mathcal{C}$ is a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ B' \arrow[r] & C \end{tikzcd}$$

[27] This implies that the diagram is equipped with a homotopy H between $p \circ i$ and b_0, and that for any space W, composition with i induces an equivalence

$$\hom(W, F) \simeq \{(g, G)\}, \qquad f \mapsto (i \circ f, H \circ f)$$

where the righthand side is the space of pairs with $g : W \to E$ continuous and G a homotopy between pg and the constant map $W \to B$ with image b_0.

such that, for any object W, the diagram of mapping spaces

$$\begin{tikzcd} \hom(W,A) \arrow[r] \arrow[d] & \hom(W,B) \arrow[d] \\ \hom(W,B') \arrow[r] & \hom(W,C) \end{tikzcd}$$

is a homotopy pullback diagram of spaces. Letting $\mathcal{C} = \mathsf{Spectra}$ and taking $W = \mathbb{S}^0$, conclude that any homotopy pullback diagram of spectra gives rise to a long exact sequence of homotopy groups.

(f) It is a fact that any homotopy pushout diagram of spectra is in fact a homotopy pullback diagram.[28] Using the fact that Σ^∞ preserves all colimits (and hence sends homotopy pushout diagrams of pointed spaces to homotopy pushout diagrams of spectra) prove that any homotopy pushout diagram of pointed spaces

$$\begin{tikzcd} U \arrow[r] \arrow[d] & W \arrow[d] \\ W' \arrow[r] & V \end{tikzcd}$$

gives rise to a long exact sequence of homotopy groups

$$\ldots \to \pi_k(\Sigma^\infty U) \to \pi_k(\Sigma^\infty W) \oplus \pi_k(\Sigma^\infty W') \to \pi_k(\Sigma^\infty V) \to \pi_{k-1}(\Sigma^\infty U) \to \ldots.$$

Remark II.17.0.2. It turns out that most good covers $V = W \cup W'$ in nature give rise to homotopy pushout diagrams by setting $U = W \cap W'$. Thus, even though Remark II.17.0.1 would discourage us from seeking Mayer-Vietoris type computations of homotopy groups of spaces, we see that *stable* homotopy groups satisfy the Mayer-Vietoris property.

II.18. Whitehead Theorem for spectra

In this exercise, we will sketch a proof of Whitehead's theorem (the version for spectra):

Proposition II.18.0.1 (Whitehead's theorem, spectrally). Let $f : X \to Y$ be a map of spectra inducing an isomorphism on all homotopy groups $\pi_i, i \in \mathbb{Z}$. Then f is an equivalence of spectra.

(a) Convince yourself that the homotopy pullback of

$$\begin{tikzcd} & Y \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

(where $*$ is the trivial spectrum, otherwise known as the zero spectrum) is Y, with the map $Y \to Y$ (homotopic to) the identity morphism.

[28] This is is more or less equivalent to the fact that $\mathsf{Spectra}$ is a *stable* ∞-category; as mentioned in the beginning of these notes, this is the one important phenomenon we aren't able to cover in these lectures.

(b) Using the long exact sequence of homotopy groups of spectra, show that if $f : X \to Y$ satisfies the hypotheses of Whitehead's theorem, then the pushout of f along the zero map $X \to *$ is the trivial spectrum.
(c) Assuming[29] that every pushout square in $\mathsf{Spectra}$ is also a pullback square, prove Whitehead's theorem.

II.19. Some simple mapping spaces

Let us say that a spectrum Y is *discrete* if $\pi_i(Y) = 0$ for all $i \neq 0$.

(a) Let X be a suspension spectrum (meaning $X = \Sigma^\infty A$ for some pointed space A), and let Y be a discrete spectrum. Show that $\hom_{\mathsf{Spectra}}(X, Y)$ has trivial homotopy groups in degrees ≥ 1.
(b) Fix an integer $k \geq 0$. Recall, or prove, that the collection of spaces whose homotopy groups vanish above degree k is closed under all homotopy limits. (Hint: It suffices to prove this that this collection is closed under all products, and homotopy fiber products. The former is easy to see, and the latter allows you to study long exact sequences of homotopy groups.)
(c) Let X be an arbitrary spectrum, and let Y be a discrete spectrum. Show that $\hom_{\mathsf{Spectra}}(X, Y)$ has trivial homotopy groups in degrees ≥ 1.
(Hint: (II.16.1) is an equivalence.)

II.20. QS^0 cannot be made commutative on the nose

(a) Suppose G is a topological abelian group. Show that for every $n \geq 0$, the set of continuous maps $\hom_{\mathsf{Spaces}}(\Delta^n, G)$ is an abelian group.
(b) Conclude[30] that G is homotopy equivalent to a simplicial abelian group (given by the singular complex $\mathsf{Sing}(G)$).
(c) Conclude[31] that G is homotopy equivalent to a direct product of Eilenberg-Maclane spaces.

Recall that any space X has a unique-up-to-homotopy-equivalence Postnikov tower $X \to \ldots \to \tau_{\leq 2}X \to \tau_{\leq 1}X \to \pi_0(X)$ where $X \to \tau_{\leq n}(X)$ is an isomorphism on π_i for $i \leq n$ and where the homotopy groups of $\tau_{\leq n}$ vanish in degrees $> n$. In particular, each projection

$$\tau_{\leq n+1}X \to \tau_{\leq n}X$$

is a fibration with fiber $K(\pi_{n+1}(X), n+1)$. Fibrations with such fibers are classified by homotopy classes of maps to

$$BK(\pi_{n+1}(X), n+1) \simeq K(\pi_{n+1}(X), n+2)$$

[29] This is the most important property of the ∞-category of spectra; it is tantamount to the *stability* of the ∞-category.

[30] Requires prior knowledge of simplicial abelian groups.

[31] Requires knowledge of the Dold-Kan correspondence, and that fact that any chain complex is equivalent, passing to a free resolution if necessary, to its homology.

where B is the classifying space construction[32]. On the other hand, homotopy classes of maps to $K(A, n+1)$ are exactly elements of $H^{n+1}(-; A)$. Thus, these towers can be understood by understanding each $\tau_{\leq n}X$ and each element

$$k_n \in H^{n+2}(\tau_{\leq n}X, \pi_{n+1}(X))$$

is called the nth k-invariant of X.

(d) Show that if X is a product of Eilenberg-MacLane spaces, all its k-invariants vanish.
(e) Let $Y = S^n$ for n large enough, so that $\pi_{n+1}(S^n) = \mathbb{Z}/2\mathbb{Z}$. (This is a consequence of Freudenthal Suspension theorem and a computation of $\pi_4(S^3)$.) Argue that the nth k-invariant $k_n \in H^{n+2}(\tau_{\leq n}S^n; \mathbb{Z}/2\mathbb{Z})$ must be non-trivial. (Hint: $H_{n+1}(S^n)$ does not equal $\mathbb{Z}/2\mathbb{Z}$.)
(f) By staring at the definition of QS^0, conclude that the k_n for S^n fit together to define a single element k_0 in the 2nd cohomology of the spectrum $H\mathbb{Z}$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$, and that k_0 must be non-trivial.
(g) Conclude that QS^0 cannot be homotopy equivalent to a product of Eilenberg-MacLane spaces.
(h) Conclude that QS^0, or any space homotopy equivalent to it, cannot be given the structure of a topological abelian group.

II.21. The sphere spectrum as a ring rears its head

(a) Show that the endomorphism space $\hom_{\mathrm{Sp}}(\mathbb{S}^0, \mathbb{S}^0)$ is equivalent to the 0th space of the sphere spectrum, otherwise known as QS^0.
(b) Show that the homotopy groups $\pi_* QS^0 \cong \pi_* \mathbb{S}^0$ form a graded abelian group.

The space $\hom_{\mathrm{Sp}}(\mathbb{S}^0, \mathbb{S}^0) \simeq QS^0$ has *two* kinds of operations – one is additive, by construction, and the other is "multiplicative," by virtue of being an endomorphism space.

(c) Show that composition of endomorphisms renders $\pi_* \mathbb{S}^0$ a graded ring.

(Hint: An element of $\pi_n QS^0$ is some map $f : S^{n+m} \to S^m$, and an element of $\pi_{n'} QS^0$ is some map $f' : S^{n'+m'} \to S^{m'}$. By suspending, we may assume $m \geq m' + n'$, and we may consider the composite

$$\Sigma^{m-m'-n'} f' \circ f : S^{n+m} \to S^m = S^{m'+n'+(m-m'-n')} \to S^{m'+(m-m'-n')}.$$

This is a map of degree $n + n'$. So you can see pretty hands-on that the sphere spectrum really is all about understanding homotopy classes of maps between spheres, and only in the "stable range" where we have suspended as many times as we want.)

[32]Such a construction exists for any $\mathbb{E}_1$-algebra in spaces; but here, you can take a model for an Eilenberg-Maclane space as a simplicial abelian group to write down an even more concrete model.

REMARK II.21.0.1. What is not yet obvious geometrically is that $\mathbb{S}$ is a commutative ring; indeed, why should an endomorphism algebra be commutative? Of course, once we know that $\mathbb{S}^0$ is the unit of a symmetric monoidal structure, it follows formally that its endomorphisms is a commutative ring.

II.22. The graded ring of homotopy groups

Recall that for a differential graded algebra A, its cohomology groups $H^*(A)$ form a graded ring. In this exercise we will convince ourselves of the following analogue for spectra:

PROPOSITION II.22.0.1. Let R be a spectrum. Suppose that there exists a map $u : \mathbb{S} \to R$ (called a unit) and a map $m : R \otimes R \to R$ (called multiplication) which is associative up to homotopy. Then m induces the structure of a graded unital ring on $\pi_* R$. If m is further commutative up to homotopy, then $\pi_* R$ is a graded commutative ring.

COROLLARY II.22.0.2. Let R, u, m be as above. Then for any i, $\pi_i(R)$ is a bimodule over $\pi_0 R$.

REMARK II.22.0.3. As you know, differential graded algebras A have far more structure than that of $H^*(A)$ – for example, Massey products. In a similar way, ring spectra will have far more subtle invariants than just their homotopy groups; but to define or detect such invariants, one must have more structure than mere knowledge that m is associative up to homotopy, or commutative up to homotopy. The data of the homotopies will be part of the definition of a ring spectrum and commutative ring spectrum.

(a) Explain every map in the composition

$$\begin{aligned} \hom(\mathbb{S}^i, R) \times \hom(\mathbb{S}^j, R) &\xrightarrow{\otimes} \hom(\mathbb{S}^i \otimes \mathbb{S}^j, R \otimes R) \\ &\simeq \hom(\mathbb{S}^{i+j}, R \otimes R) \\ &\xrightarrow{m} \hom(\mathbb{S}^{i+j}, R) \end{aligned}$$

and explain how this composition defines a map $\pi_i(R) \times \pi_j(R) \to \pi_{i+j}(R)$.

(b) Paying careful attention to the swap equivalence $\mathbb{S}^i \otimes \mathbb{S}^j \simeq \mathbb{S}^j \otimes \mathbb{S}^i$, show that this product is graded commutative if m is commutative up to homotopy.

(c) Recall that the natural map

$$X \otimes Y \oplus X' \otimes Y \to (X \oplus X') \otimes Y$$

is an equivalence, since $\otimes$ preserves coproducts in each variable. Show that for any triplet of maps a, b, c, the each square in the below diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathbb{S}^{i+j} \simeq \mathbb{S}^i \otimes \mathbb{S}^j \arrow[r, "\Delta_{\mathbb{S}^i}\otimes \mathrm{id}"] \arrow[dr, "\Delta"] & (\mathbb{S}^i \oplus \mathbb{S}^i) \otimes \mathbb{S}^j \arrow[r, "(a\oplus b)\otimes c"] \arrow[d, "\simeq"] & (R \oplus R) \otimes R \arrow[r, "+\otimes \mathrm{id}"] \arrow[dr, "\simeq"] & R \otimes R \arrow[r, "m"] & R \\
 & \mathbb{S}^i \otimes \mathbb{S}^j \oplus \mathbb{S}^i \otimes \mathbb{S}^j \arrow[rr, "a\otimes c \oplus b \otimes c"] & & R \otimes R \oplus R \otimes R \arrow[r, "m\oplus m"] & R \oplus R \arrow[u, "+"]
\end{tikzcd}$$

Here, the maps Δ are induced by suspended pinch maps (Exercise II.14), or by shifts of pinch maps (if i, j are negative, for example). (Hint: the coherence of the rightmost trapezoid requires also uses the universal property of $\oplus$ to characterize maps out of $R \otimes R \oplus R \otimes R$.)

(d) Prove the proposition.

II.23. A formula for computing the smash product

Here we present a riff on one classical model of smash product: The "handicrafted" or "naive" smash product of Boardman, which Adams also riffs on in his famous blue book[33].

Given two prespectra X and Y, one can form a bigraded sequence of spaces whose (i,j)th space is given by $X_i \wedge Y_j$. The maps $\Sigma X_i \to X_{i+1}$ and $\Sigma Y_j \to Y_{j+1}$, by adjunction, defines a $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ diagram of spaces as follows:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \cdots \\
\Omega^2(X_0 \wedge Y_2) \arrow[r] \arrow[u] & \Omega^3(X_1 \wedge Y_2) \arrow[r] \arrow[u] & \Omega^4(X_2 \wedge Y_2) \arrow[r] \arrow[u] & \cdots \\
\Omega(X_0 \wedge Y_1) \arrow[r] \arrow[u] & \Omega^2(X_1 \wedge Y_1) \arrow[r] \arrow[u] & \Omega^3(X_2 \wedge Y_1) \arrow[r] \arrow[u] & \cdots \\
X_0 \wedge Y_0 \arrow[r] \arrow[u] & \Omega(X_1 \wedge Y_0) \arrow[r] \arrow[u] & \Omega^2(X_2 \wedge Y_0) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Then the 0th space of the smash product spectrum is defined to be the colimit of this $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ diagram. Likewise, there is a diagram defined beginning with $X_i \wedge Y_j$ for $i + j = n$, and we can define the colimit of an associated diagram as the nth space of the smash product spectrum. (For example, remove the $X_0 \wedge Y_0$ corner from the above diagram, and apply one less Ω to each entry. The colimit of the resulting diagram is the 1st space of the smash product spectrum.)

Using the above models, which of the following properties of the smash product can you convince yourself of?

[33]See III.4 of "Stable homotopy and generalized homology" by J.F. Adams.

(i) The smash product is associative. (Informally, for any triplet X, Y, Z of spectra, $X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$. Less informally, one would need to provide such equivalences coherently for any k-tuple of spectra.)
(ii) The smash product is symmetric monoidal. (Informally, $X \otimes Y \simeq Y \otimes X$ for any two spectra X, Y. Less informally, one needs to exhibit such equivalences coherently for any k-tuple, for any permutation, and in conjunction with associativity.)
(iii) The sphere spectrum is the unit. (Informally, $\mathbb{S} \otimes X \simeq X$ for any spectrum X.)
(iv) Σ^∞ is symmetric monoidal. (Informally, $\Sigma^\infty(A \wedge B) \simeq \Sigma^\infty A \otimes \Sigma^\infty B$ for any two pointed spaces A, B. Less informally, one would have to prove that Σ^∞ can be equipped with many coherences making all relevant diagrams homotopy coherent.)

Remark II.23.0.1. I have made parenthetical remarks throughout. The biggest drawback of these lectures series is the omission of the definition of symmetric monoidal ∞-categories, so we cannot make precise what the above coherences are. But we hope the informal properties give the readers enough confidence that something sensible is going on. Finally, let me assure the reader that we do *not* write down all coherences by hand. One sets up a theory in which many coherences are formally deducible from basic properties of the ∞-category of spaces. (This is similar to the way in which classical category theory is truly built up of basic properties about sets.) One then compares to concrete models by arguing that the formal properties of concrete models suffice to exhibit a universal property, showing that the concrete models must indeed describe the formally deduced structures.

II.24. Another formula for smash product

(a) Recall the equivalence (II.16.1). Now, using the fact that Σ^∞ is symmetric monoidal and preserves colimits in each variable, write a formula for $X \otimes Y$. (Your "formula" will write $X \otimes Y$ as a colimit of a diagram indexed by pairs of natural numbers.)
(b) Write also a formula for a finite tuple of spectra (as a colimit indexed by tuples of natural numbers). I encourage you not to write a formula for something like $X \otimes (Y \otimes (Z \otimes W))$, but rather a formula for something like $X \otimes Y \otimes Z \otimes W$.
(c) Using the above formulas, which of the following properties of the smash product can you convince yourself of?
   (i) The smash product is associative.
   (ii) The smash product is symmetric monoidal.
   (iii) The sphere spectrum is the unit.
   (iv) Σ^∞ is symmetric monoidal.

II.25. Homology

We are positing the existence of a symmetric monoidal structure $\otimes$ on $\mathsf{Spectra}$. Let us at least give one consequence of this by giving a new construction of homology. First, recall that $H\mathbb{Z}$ is the spectrum associated to the integers (Definition I.4.0.7). Consider the following (seemingly arbitrary) composition of functors:

$$\mathbb{H}_i : \mathsf{Spaces}_* \xrightarrow{\Sigma^\infty} \mathsf{Spectra} \xrightarrow{\otimes H\mathbb{Z}} \mathsf{Spectra} \xrightarrow{\pi_i} \mathsf{Ab}.$$

Concretely, $\mathbb{H}_i$ takes a pointed space X to the group $\pi_i(\Sigma^\infty X \otimes H\mathbb{Z})$. In this exercise, I want you to prove the following:

Proposition II.25.0.1. $\mathbb{H}_i$ is naturally equivalent to the reduced homology functor $\tilde{H}_i(-,;\mathbb{Z})$ with coefficients in $\mathbb{Z}$. That is, for any pointed space X, there is an isomorphism

$$\pi_i(\Sigma^\infty X \otimes H\mathbb{Z}) \cong \tilde{H}_i(X;\mathbb{Z}).$$

(a) We will proceed by showing the collection $\{\mathbb{H}_i\}_{i\geq 0}$ satisfies the Eilenberg-Steenrod axioms for reduced homology of CW complexes. Look those up as a refresher, so you have some guidance for the strategy.
(b) In Spaces_*, consider those pointed spaces that are homotopy equivalent to CW complexes, in which case the long exact sequence axiom only needs to be checked for pushouts of spaces of the form $B \cup_A C$ where A includes as a sub-CW-complex into B and C. Such a pushout is in fact a homotopy pushout of spaces (you may take this for granted). Use Exercise II.17 to prove that long exact sequence axiom is satisfied for $\mathbb{H}_i$.
(c) Show that the dimension axiom is satisfied – that is, show that $\mathbb{H}_i(S^0)$ is isomorphic to $\mathbb{Z}$ when $i = 0$, and equal to 0 otherwise.
(d) Now suppose you have an infinite wedge sum $\vee_\alpha X_\alpha$ of spaces. Because Σ^∞ preserves colimits, we have that $\Sigma^\infty(\vee_\alpha X_\alpha) \simeq \oplus_\alpha \Sigma^\infty X_\alpha$, where the latter is an infinite direct sum.[34] It turns out that Ω^∞ commutes with infinite direct sums[35]. Using the fact that S^0 is compact, show that

$$\mathbb{H}_i(\vee_\alpha X_\alpha) \cong \oplus_\alpha \mathbb{H}_i(X_\alpha).$$

Remark II.25.0.2. The same proof shows that for any abelian group A, the functor $X \mapsto \pi_i \Sigma^\infty X \otimes HA$ is isomorphic to reduced homology with coefficients in A.

Remark II.25.0.3. If you want to recover the (unreduced) homology of a space X, simply consider the space $X_+ := X \coprod *$ obtained by adjoining a disjoint basepoint to X. Then $\mathbb{H}_i(X_+) \cong H_i(X)$.

[34] We use the "direct sum" notation again to analogize with abelian groups. It is still just an infinite coproduct, if that means something to you.

[35] And more generally, with filtered colimits.

Remark II.25.0.4. Of course, we have not yet proven that the isomorphisms are natural; they in fact are, but to prove the isomorphism, I would have to read your mind about which model of reduced homology you prefer.

One of my favorite models is singular homology. One can prove that the operation $X \mapsto H(C_*(X))$, which takes a space and maps it to the Eilenberg-Maclane spectrum associated to its singular chain complex, is naturally equivalent to the operation $X \mapsto \Sigma^\infty X \otimes H\mathbb{Z}$. Whatever natural equivalence one writes down to realize this will result in a proof that the homology theories given by $\mathbb{H}$ and $\tilde{H}$ are equivalent.

So how would we go about proving this natural equivalence? While we do not yet have the precise technology in this chapter to speak of the following method precisely, allow me to explain it anyway.

The ∞-category of chain complexes is tensored over Spaces, because it has all colimits. It turns out that the functor $X \mapsto C_*(X)$ can be modelled as the functor

$$\mathsf{Spaces} \to \mathsf{Chain}, \qquad X \mapsto \operatorname{colim}_X \mathbb{Z}.$$

That is, one takes the colimit of the constant diagram $X \to \mathsf{Chain}$ with value $\mathbb{Z}$. One often denotes $\operatorname{colim}_X \mathbb{Z}$ also as $X \otimes \mathbb{Z}$.

One can then prove that the Dold-Kan construction preserves colimits. Thus the composite of $X \mapsto C_*(X) \to HC_*(X)$ can be expressed as

$$\operatorname{colim}_X H\mathbb{Z}.$$

On the other hand, any pointed space X can be written tautologically as a colimit of S^0 indexed by the diagram X itself. Since Σ^∞ preserves colimits (being a left adjoint), we have that

$$\Sigma^\infty X \otimes H\mathbb{Z} \simeq (\operatorname{colim}_X \mathbb{S}) \otimes H\mathbb{Z} \simeq \operatorname{colim}_X(\mathbb{S} \otimes H\mathbb{Z}) \simeq \operatorname{colim}_X H\mathbb{Z}$$

where the middle equivalence follows from $\otimes$ respecting colimits in each variable.

II.26. Cohomology

We've seen that we can recover the (reduced) homology of a pointed space by smashing with an Eilenberg-Maclane spectrum.

Now let's recover cohomology. Again, fix an abelian group A and a topological space X. We will now study the spectrum of maps from $\Sigma^\infty(X_+)$ to HA. In this problem, you will prove:

Proposition II.26.0.1. For every $i \geq 0$, there exists an isomorphism of abelian groups

$$\pi_{-i} \hom_{\mathsf{Spectra}}(\Sigma^\infty(X_+), HA) \cong H^i(X; A). \tag{II.26.1}$$

(a) Show that

$$\pi_{-i} \hom_{\mathsf{Spectra}}(\Sigma^\infty(X_+), HA) \cong \pi_0 \hom_{\mathsf{Spaces}_*}(X_+, \Omega^\infty(\Sigma^i HA)).$$

(b) Show that $\Omega^\infty \Sigma^i HA = \Omega^\infty HA[i]$ is homotopy equivalent to the space

$$K(A, i).$$

(c) It is a classical fact that the Eilenberg-Maclane space $K(A,i)$ represents ith cohomology with coefficients in A:

$$\pi_0 \hom_{\mathsf{Spaces}_*}(X_+, K(A,i)) \cong \pi_0 \hom_{\mathsf{Spaces}}(X, K(A,i)) \cong H^i(X;A).$$

So conclude your proof.

Remark II.26.0.2. As you know, if A is furthermore a commutative ring, then $H^*(X;A)$ has the structure of a graded commutative ring.

How does this play with the isomorphism (II.26.1)? In the coming lectures, we will gain some familiarity with ring spectra, and we will see that for any ring R, the Eilenberg-Maclane spectrum HR is rendered a ring spectrum. At the very least, such spectra will come equipped with the data of maps

$$m : HR \otimes HR \to HR$$

where $\otimes$ is the smash product of spectra. Because the smash product is a symmetric monoidal structure on the ∞-category of spectra, will have maps

$$\hom(Y, HR) \times \hom(Y, HR) \xrightarrow{\otimes} \hom(Y \otimes Y, HR \otimes HR) \xrightarrow{m} \hom(Y \otimes Y, HR). \tag{II.26.2}$$

Now, when $Y = \Sigma^\infty(X_+)$ arises as the suspension spectrum of an unpointed space X, we have the equivalence

$$Y \otimes Y \simeq \Sigma^\infty((X \times X)_+) \tag{II.26.3}$$

because Σ^∞ is a symmetric monoidal functor. But we of course have the diagonal map $X \to X \times X$, which induces a map $X_+ \to (X \times X)_+$, and hence a map

$$\Sigma^\infty(X_+) \to \Sigma^\infty((X \times X)_+). \tag{II.26.4}$$

Pulling back along this suspended diagonal map, composition with (II.26.2) and (II.26.3) induces a map

$$\hom(\Sigma^\infty(X_+), HR) \times \hom(\Sigma^\infty(X_+), HR) \to \hom(\Sigma^\infty(X_+), HR). \tag{II.26.5}$$

And the ∞-category of spectra is enriched over itself (this was hinted at in Exercise I.17); so this map of spaces actually lifts to a map of spectra

$$\hom_{\mathbb{S}}(\Sigma^\infty(X_+), HR) \otimes \hom_{\mathbb{S}}(\Sigma^\infty(X_+), HR) \to \hom_{\mathbb{S}}(\Sigma^\infty(X_+), HR).$$

It turns out that this map (with higher coherence data) renders $\hom_{\mathbb{S}}(\Sigma^\infty(X_+), HR)$ a ring spectrum.

It is also a fact that ring spectra define graded rings by passing to their homotopy groups – see Exercise II.22. And when HR is commutative, or $\mathbb{E}_\infty$, this ring will be a graded commutative ring.

This is one manifestation of the graded commutative ring structure on cohomology with coefficients in HR.

Remark II.26.0.3. For a reader who is not comfortable with the enrichment of $\mathsf{Spectra}$ over itself, one may shift HR and compute π_0 of both sides of (II.26.5) to witness that the homotopy groups (positive and negative) of $\hom(\Sigma^\infty(X_+), HR)$ indeed form a graded ring (commutative when R is).

II.27. Steenrod operations

We saw in the previous exercise that – for any abelian group A – the homotopy groups of the mapping spectrum

$$\hom_{\mathbb{S}}(\Sigma^\infty X_+, HA)$$

define the cohomology groups of A. Let us note the following: the mapping space $\hom_{\mathsf{Spectra}}(HA, HA)$ has a product called composition. This turns out to not be very interesting, but the mapping spectrum

$$\hom_{\mathbb{S}}(HA, HA)$$

also has a product called composition; this renders this mapping spectrum an $\mathbb{E}_1$, or A_∞-ring in spectra by formalities involving the enrichment of this ∞-category.

Thus, there is a natural action

$$\hom_{\mathbb{S}}(\Sigma^\infty X, HA) \otimes \hom_{\mathbb{S}}(HA, HA) \to \hom_{\mathbb{S}}(\Sigma^\infty X_!, HA)$$

(a) Convince yourself that this action renders $\hom_{\mathbb{S}}(\Sigma^\infty X_+, HA)$ a right module over $\hom_{\mathbb{S}}(HA, HA)$, at least up to homotopy.
(b) Just as rings in spectra give rise (via homotopy groups) to rings in graded abelian groups, convince yourself that a module action of spectra gives rise to a module action among graded abelian groups.

So what we have seen is that there is a graded ring (it is not commutative)

$$\pi_* \hom_{\mathbb{S}}(HA, HA), \qquad * \in \mathbb{Z}$$

which acts on the graded abelian group

$$\pi_* \hom_{\mathbb{S}}(\Sigma^\infty X_+, HA) \cong H^{-*}(X).$$

Aren't you *dying* to know what this amazing ring action is? When $A = \mathbb{F}_p$, these are the mod-p Steenrod operations on the cohomology of a space.

LECTURE III

Operads

I racked my brain a good deal about how much to include in this course. My plan may be to teach you only about the classical definition of operads[1]. While pedagogically this is clearly the right thing to do, it also pains me to do so.

Why the pain? As we'll see in the exercises, the usual definition of operads actually suffer from the very problem they try to fix: They rely on certain compositions being associative on the nose, and (more generally) they ultimately rely on homotopy coherences being expressed through very particular equalities.

If there is time, I might indicate an ∞-style definition of operads; i.e., ∞-operads. This would be a do-or-die mission – I'd go for the gold, but there's a real chance of spectacular failure if time expires. The set-up for this would require some model, and the most appealing (for pedagogical purposes) would involve three new ideas – ∞-categories, the Grothendieck construction (and Grothendieck fibrations in general; this corresponds to the theory of co/Cartesian fibrations in simplicial language), and the category of finite pointed sets[2]. There is also the theory of ∞-operads developed using the language of dendroidal sets[3]. Who knows where these weeks will lead us.

For now, let me say that operads as I present them here contain the seeds of a "right way" to conceive of the ideas they are meant to capture, just as the notion of associativity[4] is certainly a right place to begin for us mere mortals before we begin to understand associative-up-to-homotopy structures. Just having the analogy of "associative versus associative-versus-homotopy" in mind will give you a good feel of how more modern homotopical uses of operads should go.

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[1]Due to May, and with predecessors in work of Adams, Mac Lane, Boardmann and Vogt.

[2]Relevant details are in Chapter 2 of Jacob Lurie's Higher Algebra, which is available on his website.

[3]The concept was introduced by Moerdijk and Weiss. As of this writing, it is a recent announcement due to Moerdijk and Hinich that one can prove an (expected) equivalence between the theory of ∞-operads due to Lurie and the theory of ∞-operads using dendroidal set. Indeed, this is the most recent pre-print on the arXiv with Moerdijk as an author (uploaded June 2022).

[4]I think it's under-appreciated that whoever isolated "associativity" as an important property of arithmetic operations advanced the field of mathematics greatly.

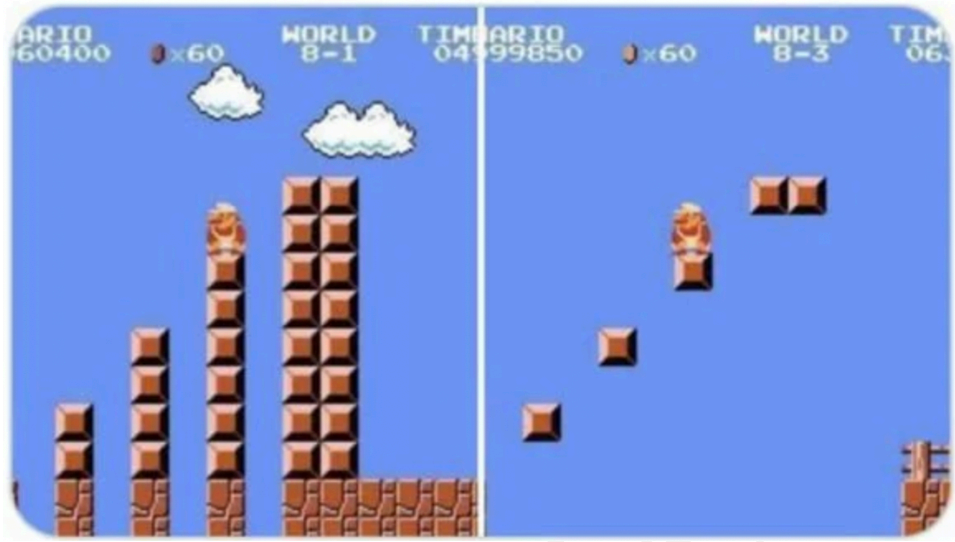

FIGURE III.0.0.1. Even when the things we must do in two situations are equivalent, additional structures can make one situation far easier than another.

One final remark. There was a meme at one point (Figure III.0.0.1), consisting of a side-by-side image of the end of two levels in Super Mario Brothers. "Physically," what one must accomplish in these two levels is identical. Thus, the meme claimed, there is no difference in these two levels, and the perceived difficulty of one over the other is "all psychological" (and not "real"). But the perception of difficulty is real. What one level provides – an array of blocks and lines to help frame the spatial relations between targets – makes the level far easier to navigate than the other. In a similar way, operads can provide lines that help us better understand our targets, and improve the way in which we perceive the structures around us.

III.1. A motivating example: The $\mathbb{E}_n$ operad

Let me give an idea of what operads are supposed to encode, via example.

Recall that the n-fold loop space $\Omega^n X$ is supposed to be a group that is "commutative up to an $(n-1)$-dimensional sphere." In other words, each time you try to write an equality of the form $\gamma\gamma' = \gamma'\gamma$, one must know that this expression becomes ambiguous if we try to write it a sphere's worth of times.[5] Thus, there seems to be some algebraic structure that is not quite

[5] This is analogous to the role of curvature in differential geometry. Covariant derivatives do not commute, but each time you want to make them commute, you must sacrifice a curvature term. Likewise, each time you pretend that two elements of $\Omega^n X$ commute, you can do so up to the cost of some higher-degree term, and the non-zeroness of this term only becomes apparent if you try to integrate if over S^{n-1}.

commutative, but not just associative. One would like a language for such (and more general kinds of) algebraic structures.

The starting point is the realization that elements of $\Omega^n X$ may be composed in a space of standard ways.

Definition III.1.0.1 (Rectilinear embeddings). An embedding $j : (0,1)^n \to (0,1)^n$ is called *rectilinear* if $j(x) = Ax + v$, where $v \in (0,1)^n$ and A is a diagonal matrix whose entries are positive real numbers.[6]

Given a finite disjoint union $\coprod_k (0,1)^n$ of n-dimensional cubes, an embedding $j : \coprod_k (0,1)^n \hookrightarrow (0,1)^n$ is called rectilinear is j is rectilinear on each component of the domain.

Notation III.1.0.2 ($\mathbb{E}_n(k)$). We let $\mathbb{E}_n(k)$ denote the space of all rectilinear embeddings $\coprod_k (0,1)^n \hookrightarrow (0,1)^n$.

Let's fix $n \geq 0$. (You can take $n = 2$ for concreteness, if you like.) You will have the occasion to explore the spaces $\mathbb{E}_n(k)$ a bit in the exercises, so let's black box what they look like for a bit.

What occupies us now is that the collection of spaces $\mathbb{E}_n(0), \mathbb{E}_n(1), \mathbb{E}_n(2), \ldots$ admit extra structures that renders it an *operad*. These are the structures of composition, and of symmetric group actions.

Construction III.1.0.3 (Symmetric group actions). For each $k \geq 0$, $\mathbb{E}_n(k)$ has an action by the symmetric group on k letters, by acting on the connected components of $\coprod_k (0,1)^n$.

Construction III.1.0.4 (Composition). Fix an integer $j \geq 0$, and fix integers $i_1, \ldots, i_j \geq 0$. Then there is a map

$$\mathbb{E}_n(j) \times (\mathbb{E}_n(i_1) \times \mathbb{E}_n(i_2) \times \ldots \times \mathbb{E}_n(i_j)) \to \mathbb{E}_n(i_1 + \ldots + i_j). \tag{III.1.1}$$

In words: if I have a single embedding of j medium cubes into a big cube, and if for all $1 \leq a \leq j$, I have an embedding of i_a many small cubes into the ath medium cube, I can compose all of these embeddings to obtain an embedding of $i_1 + \ldots + i_j$ many small cubes into the big cube.[7]

There should be a picture here. If you want to hand-draw a nice picture, please let Hiro know and send it over.

Remark III.1.0.5. There is more to be said. For example, the composition map (III.1.1), as one varies $j, i_1, \ldots, i_j$, are associative on the nose. Likewise, the composition map is equivariant with respect to the $\Sigma_j \times \Sigma_{i_1} \times$

[6] One could also take A to simply be a real positive scalar; though the space of such rectilinear embeddings is not homeomorphic to the one defined in the main text, it is homotopy equivalent. We use the definition in the main text as it follows historical convention, and it also includes the usual embeddings we use to define the group operations of π_n.

[7] The cubes $(0,1)^n$ of course have the same size; but I hope the adjectives help more than confuse.

$\ldots \times \Sigma_{i_j}$ actions on both domain and codomain. There are also two senses in which these data have a unit.

For the sake of time, most of these properties – which are *checks* of whether the data of Constructions III.1.0.3 and III.1.0.4 satisfy the properties, and not extra data – will be relegated to the exercises (if they are covered at all).

III.2. Definition of operads and examples

DEFINITION III.2.0.1 (Operads). An *operad* $\mathcal{O}$ is the data of

(1) For every integer $k \geq 0$, a space $\mathcal{O}(k)$. We call this the k-ary operation space of $\mathcal{O}$.
(2) For every k, a continuous Σ_k action on $\mathcal{O}(k)$.
(3) For every integer $j \geq 0$ and every collection of integers $i_1, \ldots, i_j \geq 0$, a composition map

$$\mathcal{O}(j) \times \mathcal{O}(i_1) \times \ldots \times \mathcal{O}(i_j) \to \mathcal{O}(i_1 + \ldots + i_j).$$

These must satisfy associativity, equivariance, and unitality[8] conditions that were glossed over in Remark III.1.0.5.

EXAMPLE III.2.0.2 (The little n-disks operad $\mathbb{E}_n$). Fix $n \geq 0$. Then the spaces $\mathbb{E}_n(k)$, along with their symmetric group actions and composition maps, form an operad. We denote this operad by $\mathbb{E}_n$. It goes by the names of the little n-cubes operad, the little n-disks operad, and the $\mathbb{E}_n$-operad.

EXAMPLE III.2.0.3 (The endomorphism operad). Let X be a topological space. Then for every $k \geq 0$, consider the space

$$\mathcal{E}\mathrm{nd}_X(k) := \hom_{\mathsf{Spaces}}(X \times \ldots \times X, X)$$

where the product is taken k times. (When $k = 0$, $X^0 = \emptyset$; hence $\mathcal{E}\mathrm{nd}_X(0)$ is a one-point space.) There is an obvious Σ_k-action on each $\mathcal{E}\mathrm{nd}_X(k)$ given by permuting the factors in X^k. Composition is defined by taking

$$(g : X^j \to X, f_1 : X^{i_1} \to X, \ldots, f_j : X^{i_j} \to X) \mapsto g \circ (f_1 \times \ldots \times f_j).$$

Thus, operads exist in abundance.

In fact, for any symmetric monoidal category, and for any such thing where one can give a natural space enrichment (meaning the morphism sets can be thought of as morphism *spaces* in a natural way) the endomorphism operad is an operad in spaces.

EXAMPLE III.2.0.4 (The commutative operad). We let $\mathcal{C}\mathrm{omm}$ denote the operad for which

$$\mathcal{C}\mathrm{omm}(k) := *$$

[8] Be aware that many people use operads that are not unital, just as many people study and use algebras that are not unital; such non-unital things just show up sometimes.

for any $k \geq 0$. That is, this operad's k-ary space is just (the space consisting of) one point. This uniquely defines all the symmetric group and composition operations. We call this $\mathcal{C}\mathrm{omm}$ the *commutative operad*; we'll see why soon.

Example III.2.0.5 (The $\mathbb{E}_\infty$ operad). Note that for every $n, k \geq 0$, we have a natural map

$$\mathbb{E}_n(k) \to \mathbb{E}_{n+1}(k).$$

This map takes every embedding of n-dimensional cubes j and takes its direct product with $\mathrm{id}_{[0,1]}$, to obtain an embedding of $(n+1)$-dimensional cubes which "acts as the identity" on the $(n+1)$st coordinate. This map is clearly continuous and equivariant with respect to the Σ_k action; I promise it's also straightforward to see that this defines a map of operads $\mathbb{E}_n \to \mathbb{E}_{n+1}$. As a result, we can conclude that the increasing union, also known as the colimit spaces

$$\mathbb{E}_\infty(k) := \mathrm{colim}(\mathbb{E}_0(k) \to \mathbb{E}_1(k) \to \mathbb{E}_2(k) \to \ldots)$$

assemble to form an operad in spaces. We call the resulting operad the $\mathbb{E}_\infty$ *operad*.

Concretely, $\mathbb{E}_\infty(1)$ is the space of ways to embed the cube[9] $[0,1]^\infty$ into itself in a rectilinear fashion. A rectilinear embedding is a map of the form

$$x \mapsto Ax + v$$

where A is a $\mathbb{N} \times \mathbb{N}$ diagonal matrix, all of whose entries are positive, and only finitely many entries are non-1. v is a vector in $[0,1]^\infty$. Likewise, $\mathbb{E}_\infty(k)$ is the space of embeddings

$$[0,1]^\infty \coprod \ldots \coprod [0,1]^\infty \to [0,1]^\infty$$

which, on each component of the domain, is rectilinear.

Example III.2.0.6 (Operads in chain complexes, or other symmetric monoidal categories). More generally, suppose we have some symmetric monoidal category $\mathcal{C}^\otimes$. For concreteness, we will assume $\mathcal{C}$ is the category of chain complexes over a base ring R with symmetric monoidal structure given by $\otimes_R$.

Then an *operad* (of chain complexes) is the data of

(1) For every integer $k \geq 0$, a chain complex $\mathcal{O}(k)$.
(2) For every k, a Σ_k action on $\mathcal{O}(k)$.
(3) For every integer $j \geq 0$ and every collection of integers $i_1, \ldots, i_j \geq 0$, a composition map

$$\mathcal{O}(j) \otimes_R \mathcal{O}(i_1) \otimes_R \ldots \otimes_R \mathcal{O}(i_j) \to \mathcal{O}(i_1 + \ldots + i_j).$$

[9] This is *not* the Hilbert cube, in that this does not have the infinite-direct-product topology. Rather, this is a subspace of the vector space $\mathbb{R}^\infty \cong \oplus_{\mathbb{N}}\mathbb{R}$. So an element of $[0,1]^\infty$ only has finitely many non-zero coordinates.

These must satisfy associativity, equivariance, and unitality conditions.

An important example is the endomorphism operad for a chain complex. When V is a chain complex, R is a base ring, then for any $k \geq 0$, there is a hom *chain complex* called $\hom(V^{\otimes_R k}, V)$. This is a far more interesting object then, say, the operad in sets of chain maps $V^{\otimes k} \to V$.

In general, for any symmetric monoidal category $\mathcal{C}^{\otimes}$ it makes sense to say what an operad in $\mathcal{C}$ is. Note that for any $\mathcal{C}^{\otimes}$ and any object X in $\mathcal{C}^{\otimes}$, the endomorphism operad of X is by default an operad in sets; but if $\mathcal{C}^{\otimes}$ is self-enriched[10] then one can define a version of $\mathcal{E}\mathrm{nd}_X$ as an operad in $\mathcal{C}^{\otimes}$.

Example III.2.0.7. Suppose $\mathcal{O}$ is an operad in spaces. Then the singular chains functor allows us to define an operad in chain complexes, which one might denote by $C_*\mathcal{O}$. Concretely, the k-ary operation complex

$$C_*(\mathcal{O})(k) := C_*(\mathcal{O}(k))$$

is defined to be the singular chains on the space $\mathcal{O}(k)$. This complex inherits an obvious symmetric group action, and the Eilenberg-Zilberg map[11] induces the composition maps. This construction produces an operad with singular chains with coefficients in any base ring (where the output is an operad in the category of chain complexes over that base ring).

As an example, the chains on the little disks operad $C_*\mathbb{E}_n$ defines an operad in chain complexes. This operad is complicated in general, but when the base ring is a field of characteristic 0, a famous theorem of Kontsevich and Tamarkin asserts that it can be completely understood in terms of its cohomology.

III.3. Algebras over operads

We motivated $\mathbb{E}_n$ by studying the space of ways in which one naturally wants to compose elements in an n-fold loop space $\Omega^n X$. In fact, given the (somewhat curated) examples of operads we have seen, it would be crazy of us not to think about the following:

Construction III.3.0.1. For every element j of $\mathbb{E}_n(k)$, there is a map

$$\rho_j : \Omega^n X \times \ldots \times \Omega^n X \to \Omega^n X$$

(where there are k copies of $\Omega^n X$ in the domain). For example, Figure I.1.1.5 depicts this map for a particular j in the $n = 2, k = 2$ case. These maps depend continuously on j, so we conclude there are continuous maps

$$\rho : \mathbb{E}_n(k) \to \hom_{\mathsf{Spaces}}(\Omega^n X \times \ldots \times \Omega^n X, \Omega^n X), \qquad j \mapsto \rho_j.$$

In other words, we have for every $k \geq 0$, continuous maps

$$\mathbb{E}_n(k) \to \mathrm{End}_{\Omega^n X}(k). \tag{III.3.1}$$

[10]The chief example for us will be when $\mathcal{C}$ is the category of chain complexes over a field k and the symmetric monoidal structure is $\otimes_k$.

[11]For two spaces X and Y, this map is a quasi-isomorphism $C_*(X) \otimes_{\mathbb{Z}} C_*(Y) \to C_*(X \times Y)$.

This looks very much like it ought to encode "a map of operads." Any good definition of an object in math should tell you what maps of such objects respect. So if we believe that operads are defined well (they are), we can take the hint: Whatever a map of operads is, it should respect composition, units, and the symmetric group action.

DEFINITION III.3.0.2. Let $\mathcal{O}$ and $\mathcal{O}'$ be two operads. A map $f : \mathcal{O} \to \mathcal{O}'$ of operads is the data of continuous maps $f_k : \mathcal{O}(k) \to \mathcal{O}'(k)$ for every $k \geq 0$, such that f_k is Σ_k-equivariant, and such that f_k respects compositions and units. To be explicit, "respecting composition" means that the diagram

$$\begin{tikzcd}
\mathcal{O}(j) \times \mathcal{O}(i_1) \times \ldots \times \mathcal{O}(i_j) \arrow[r] \arrow[d] & \mathcal{O}(i_1 + \ldots + i_j) \arrow[d] \\
\mathcal{O}'(j) \times \mathcal{O}'(i_1) \times \ldots \times \mathcal{O}'(i_j) \arrow[r] & \mathcal{O}'(i_1 + \ldots + i_j)
\end{tikzcd}$$

commutes on the nose.

EXAMPLE III.3.0.3. The collection of maps (III.3.1) for $k \geq 0$ is a map of operads. For example, in the case $k = 2$, the map

$$\rho : \mathbb{E}_n(2) \to \hom_{\mathsf{Spaces}}(\Omega^n X \times \Omega^n X, \Omega^n X), \qquad j \mapsto \rho_j.$$

is in fact equivariant with respect to the swap map – given by Remark III.1.0.3 in the domain, and by swapping the factors of $\Omega^n X \times \Omega^n$ X in the target of ρ.

REMARK III.3.0.4. You may have wondered where the symmetric group actions become important or useful. Here you go: Whether the maps ρ_j are "commutative" operations can be detected purely by understanding the spaces $\mathbb{E}_n(2)$: Is the swap map of $\mathbb{E}_n(2)$ homotopic to the identity map of $\mathbb{E}_n(2)$? (Exercise: It is.) Then $\rho_{\mathrm{swap}\,j}$ is homotopic to ρ_j.

Our arguments from Lecture I about why n-fold loop spaces are "commutative up to some spheres" were very universal. They applied to all n-fold loop spaces, because at the end of the day our argument sonly depended on configuration spaces of cubes in cubes. One reason this definition of operads makes us feel "good" is that the definition separates the structures in a way commensurate with this philosophical reason for the "non-canonical abelianness" of n-fold loop spaces.

And now we can give the definition which, in so many ways, motivates the entire invention of operads.

DEFINITION III.3.0.5 (Algebras over operads). Let $\mathcal{O}$ be an operad and X an object in some symmetric monoidal category $\mathcal{C}^{\otimes}$. Then an $\mathcal{O}$-*algebra sturcture on* X is a map of operads

$$\mathcal{O} \to \mathcal{E}\mathrm{nd}_X\,.$$

When X is given a $\mathcal{O}$-algebra structure, we say that X is an $\mathcal{O}$-*algebra.*

More generally, if $\mathcal{O}$ is an operad in a category $\mathcal{V}$ and $\mathcal{C}^{\otimes}$ is a symmetric monoidal $\mathcal{V}$-enriched category, then a map of operads in $\mathcal{V}$ $\mathcal{O} \to \mathcal{E}\mathrm{nd}_X$ is a $\mathcal{O}$-algebra structure on X.

Example III.3.0.6. You will see in the exercises that a $\mathcal{C}$omm-algebra structure on X is precisely a commutative algebra structure on X. In particular, if $\mathcal{C}^{\otimes}$ is the category of vector spaces over a base field k with symmetric monoidal structure $\otimes_k$, then a $\mathcal{C}$omm-algebra structure on X endows X with a unital commutative product.

Example III.3.0.7. For any pointed topological space X, $\Omega^n X$ is an $\mathbb{E}_n$-algebra. Note this is an example of a $\mathcal{V}$-enriched notion, where $\mathcal{V}$ is the category of spaces with symmetric monoidal structure given by direct product.

Example III.3.0.8. In the exercises we will encounter an operad in vector spaces called the Lie operad, $\mathcal{L}$ie. This is an operad that has no analogue in sets or in spaces. As you might expect, a $\mathcal{L}$ie-structure on a vector space X is precisely a Lie algebra structure on X.

This is actually the first sign that (as you might already know) interesting algebraic structure arise in the linear setting that do not arise in a non-linear (e.g., set- or space-) setting.

Example III.3.0.9. If $f : \mathcal{O} \to \mathcal{P}$ is a map of operads, then f gives a way to think of any $\mathcal{P}$-algebra as an $\mathcal{O}$-algebra. For example, any $\mathbb{E}_{n+1}$-algebra is an $\mathbb{E}_n$-algebra.

Remark III.3.0.10. You have seen in Kate's talks operations between vector spaces that do not only take k inputs and output one output; there are natural operations (e.g., in string topology, and in other contexts) that take in k inputs and output l outputs for $l \geq 1$. Operads cannot not encode such "multi-output" operations.[12] An efficient language for encoding operations allowing for multiple inputs and outputs is the language of PROPs, which we won't go into. (A PROPerad, which Kate also mentioned, is a special case of PROPs.)

III.4. The issue with these definitions

How disconcerting would it be to have a lecturer give you a bunch of definitions, then tell you they're not good enough? Prepare to be disconcerted.

Already in these lectures we've had a philosophy that homotopy equivalence is equivalence. Let me give at least one way in which this principle is violated by any theory of operads that relies naively on the definitions we've given.

[12]Co-operads can encode multi-output operations, but only for 1 input; co-operads help articulate things like associative coalgebras, for example.

EXAMPLE III.4.0.1 (Equivalence of operads does not give rise to an equivalence on the collection of algebras). It is incredibly natural – and correct – to say that a map of two operads in spaces

$$\mathcal{O} \to \mathcal{P}$$

is an equivalence (of operads in spaces) if the maps $\mathcal{O}(k) \to \mathcal{P}(k)$ are homotopy equivalences for each k. As an example, consider the operad $\mathbb{E}_\infty$. It is known that each k-ary space of $\mathbb{E}_\infty$ is contractible – see Exercise III.23. Moreover, there is a map[13]

$$\mathbb{E}_\infty \to \mathcal{C}\mathrm{omm}\,.$$

Because $\mathbb{E}_\infty(k)$ is contractible for every k, it must be that this map is an equivalence.

Then, in any reasonable universe, one would expect that any $\mathbb{E}_\infty$-algebra should be an example of a commutative algebra. In other words, if any space X admits a map of operads

$$\mathbb{E}_\infty \to \mathcal{E}\mathrm{nd}_X,$$

there must surely be some map of operads

$$\mathcal{C}\mathrm{omm} \to \mathcal{E}\mathrm{nd}_X$$

(because E_∞ is equivalent to $\mathcal{C}\mathrm{omm}$). But it turns out that this simply isn't true. For example, the space QS^0 (or any space homotopy equivalent to it) cannot be given the structure of a commutative group in spaces (Exercise II.20). In contrast, QS^0 – and any space arising as the 0th space of a suspension spectrum – can be given the structure of an $\mathbb{E}_\infty$ algebra (Exercise III.24).

Indeed, even though we call $\mathbb{E}_\infty \to \mathcal{C}\mathrm{omm}$ an equivalence, there's no hope of having an inverse map, even up to homotopy. Every k-ary space of $\mathcal{C}\mathrm{omm}$ has a trivial Σ_k action. But the k-ary spaces of $\mathbb{E}_\infty$ have free Σ_k actions. A

To have any hope of having a good notion of equivalences compatible with the notion of homotopy equivalence, we must have a homotopical notion of "Σ_k-equivariant" to even have an inverse "map of operads."

The underlying issue, of course, is a simple one. Throughout this theory of operads – whether it be defining the composition structures, or in defining maps of operads – we demanded that certain commutative diagrams are commutative on the nose. (This includes the "equivariance" condition, which of course can be expressed as certain diagrams involving group actions commute.) We demanded equalities.

I hope this, by now, feels unnatural. For example, suppose I asked you whether a three-fold composition of loops is "equal." Concretely: Take two

[13] In fact, there is a map from any operad $\mathcal{O}$ to $\mathcal{C}\mathrm{omm}$ because $\mathcal{C}\mathrm{omm}$ has trivial operation spaces – see Exercise III.16.

triplets of loops $\gamma_i, \gamma_i' : [0,1] \to X$ all based at some x_0. Does it make sense for me to ask

$$\text{“Does } \gamma_3 \sharp \gamma_2 \sharp \gamma_1 \text{ equal } \gamma_3' \sharp \gamma_2' \sharp \gamma_1' \text{?”}$$

Well, to even pose the question, you'd want me to specify how I've divided up the interval $[0,1]$ to define the concatenation of three loops, but all along you'd also know such a division would be an arbitrary choice. A far more natural question would be to ask whether the triple compositions are homotopic; and even better than a question, a good *system* for organizing this data would be to specify all the reasonable/natural compositions and remembering the various homotopies between them.

This is what the $\mathbb{E}_1$ operad succeeds in articulating, but what the definition of operads fails to incorporate. To put it in slightly vague terms, the theory of operads lives over the idea of the associative operad, and not over the idea of the $\mathbb{E}_1$ operad.[14]

Let me conclude the lecture by saying that we have ways of dealing with these issues. A traditional way is to use the theory of model categories – roughly speaking, you declare that there are “good” kinds of operads[15] where equivalences behave well. A notable requirement for being good is for the Σ_k actions to be free.

Another, more recent way is to reframe everything in the language of ∞-operads. This is similar to the way in which we secretly constructed a satisfying framework for smash product using the language of ∞-categories. I might have you believe that adding “∞-” to everything solves all your issues, but that is a linguistic con. The real solution to your issue is creating an efficient system that encodes both the ability and the data to handle homotopy coherences. The underlying strength of ∞-language is that we have concrete, manageable *combinatorial*[16] definitions of such ideas, and these definitions turn out to be highly amenable to categorical applications.

Remark III.4.0.2. If you are knee-deep in the way that geometers use A_∞-algebras and A_∞-categories, there is another big issue. The model that you use for the A_∞-operad is fine, but the definition that we give in this talk for a map of A_∞-algebras will *not* recover the notion of a map of A_∞-algebras. That is not your fault, of course, but these definitions' fault.

[14]This can probably be made precise using the language of multicategories – we'd prefer operads to model multicategories over $\mathbb{E}_1$, not over $\mathcal{A}ss$. But we are still using too-classical language to even articulate this notion. (The ideas are fine, the words are clunky.)

[15]Once you choose a model structure on the category of operads, these are the operads that are simultaneously cofibrant and fibrant.

[16]Just to give you an idea of the flavor of combinatorics, I will claim to you that we could prove everything we have spoken about in these lectures without ever defining the notion of a topology, so long as we understand that everything I call a “topological space” is defined combinatorially, just as simplicial complexes can be described purely combinatorially. This is the power of *simplicial sets* – Definition VI.2.0.8.

In the setting of stable things (i.e., things with shifts) like chain complexes and spectra, there is a toolkit called *Koszul duality* that actually give a formal way to try and write formulas that give rise to "the right thing." In particular, it turns out that the map of A_∞-algebras one uses in Floer theory can be recovered as a map between Koszul dual coalgebras.

III.5. For next time

What did I do today? By analogy, I told you the notion of associativity, and then concluded by telling you associativity-up-to-homotopical-data is better. Literally, I told you about the definition of operads, and told you why you should have in mind that more homotopically flexible notions are better. I did not tell you how to define such more flexible things, but did try to assure you of their existence.

Next time, I will stop ignoring the fact that this is a summer school with "Floer" in the title. I will try to exposit some of the operads that show up commonly, at least when studying Lagrangian Floer theory, or Reeb dynamics, in a setting with no holomorphic spheres.

III.6. Historical success: Cohen's computations (not covered in lecture)

Fix a base ring, which we'll assume to be a field to simplify some formulas. Given an operad $\mathcal{O}$ in spaces, one can compute $H_*(\mathcal{O}(k))$ for all k. Then if A is an $\mathcal{O}$-algebra, the maps

$$H_*(\mathcal{O}(k)) \otimes H_*(A) \otimes \ldots \otimes H_*(A) \to H_*(A)$$

allows one to write down multilinear operations on the homology of A.

Assuming A is a space, one has a diagonal map $A \to A \times \ldots \times A$ which is Σ_k equivariant with respect to the trivial action on the domain (and the permuting action on the codomain). Thus, the composition

$$\mathcal{O}(k) \times A \to \mathcal{O}(k) \times A \times \ldots \times A \to A$$

gives rise to interesting operations after computing equivariant homology of the domain. (When $\mathcal{O}(k)$ has a free action by Σ_k – for example, when $\mathcal{O} = \mathbb{E}_n$ or $\mathbb{E}_\infty$, note that the honest Σ_k quotient of the domain is already a Borel construction for the domain.)

At the time, people understood that spectra (or, more concretely, the 0th spaces of spectra) allowed for operations on homology and cohomology with lots of structure. It came to be understood, through a combination of May and Boardman-Vogt's works, that these operations arose from the fact that infinite loop spaces have an action of $\mathbb{E}_\infty$. So it was natural to look for analogous structure in n-fold loop spaces – for their own right, and also to potentially understand the behavior as $n \to \infty$ to better understand infinite loop spaces.

REMARK III.6.0.1. There were predecessors of this idea; Kudo-Araki in 1956, and Dyer-Lashof in 1962 (all inspired by Steenrod's operations). But I must admit I am not at all an expert on this topic; I just thought it would be good to tell you some of the ideas that were floating around at the time. For context, the work of Cohen-Lada-May on this subject was published in 1976 as a book, so this was all happening as Adams was giving his Weyl lectures at the IAS about infinite loop spaces. (We saw a quote from the resulting book by Adams in Lecture I.)

Fred Cohen took on this task.

It is my understanding that Fred Cohen pioneered, and perhaps even eliminated problems in, the field by providing tour-de-force computations to understand these operations. Let me give an excerpt from his work in Figure III.6.0.2. As suggested by his confident "This is the case," he closed the door on figuring out a satisfying way to organize these operations. He opened the door to utilizing them.

THE HOMOLOGY OF $\mathcal{C}_{n+1}$-SPACES, $n \geq 0$

Fred Cohen

The construction of homology operations defined for the homology of finite loop spaces parallels the construction of homology operations defined for the homology of infinite loop spaces, with some major differences. To recall, the operations for infinite loop spaces are defined via classes in the homology of the symmetric group, Σ_j. Working at the prime 2, Browder [24], generalizing and extending the operations of Araki and Kudo [1], found that an appropriate skeleton of $B\Sigma_2$ may be used to describe natural operations which allow computation of $H_*(\Omega^{n+1}\Sigma^{n+1}X;\mathbb{Z}_2)$ as an algebra. Dyer and Lashof [8], using similar methods, subsequently obtained partial analogous results at odd primes. However, comparison of the results of Dyer and Lashof to Milgram's computation [20] of $H_*(\Omega^{n+1}\Sigma^{n+1}X)$ as an algebra made it apparent that the skeleton of $B\Sigma_p$ intrinsic to the geometry of the finite loop space failed to give sufficient operations to compute $H_*(\Omega^{n+1}\Sigma^{n+1}X;\mathbb{Z}_p)$. To be precise, only $1/p-1$ times the requisite number of operations (defined in this paper) may be described using the methods of Dyer and Lashof.

In addition, there is a non-trivial unstable operation in two variables, λ_n, which was invented by Browder; the method of using finite skeleta of $B\Sigma_p$ does not lend itself to finding the relationships between λ_n and the other operations. These relationships are especially important in determination of fine structure and our later work in IV.

An alternative method for defining operations seemed to be provided by the composition pairing and the possibility that $H_*(\Omega^{n+1}S^{n+1};\mathbb{Z}_p)$ is universal for Dyer-Lashof operations. If $n = \infty$, this method works in principle, but it fails almost completely if $n < \infty$: It is shown in IV §6 that there are finite loop spaces with many non-trivial Dyer-Lashof operations for which the composition pairing is trivial.

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The observation of Boardman and Vogt that the space of little $(n+1)$-cubes acts on $(n+1)$-fold loop spaces, together with May's theory of iterated loop spaces [G], led one to expect that the equivariant homology of the little cubes ought to enable one to define all requisite homology operations in a natural setting analogous to that provided for an infinite loop spaces by $B\Sigma_p$. This is the case. In addition, one can describe easily understood constructions with the little cubes which, when linked with May's theory of operads, enable one to determine the commutation relations between all of the operations, and between the operations and the product, coproduct, and Steenrod operations on the homology of iterated loop spaces.

Knowledge of this fine structure is essential, for example, in the analysis of the composition pairing and the Pontrjagin ring $H_*(SF(n+1);\mathbb{Z}_p)$ for all n and p in IV. Indeed, all of the formulas in III. 1.1-1.5 are explicitly used there. A further application of the fine structure is an improvement [28] of Snaith's stable decomposition for $\Omega^{n+1}\Sigma^{n+1}X$ [25].

We have tried to parallel the format of I as closely as possible, pointing out essential differences. Sections 1-4, which are analogous to I. 1,2,4, and 5, contain the computations of $H_*\Omega^{n+1}\Sigma^{n+1}X$ and $H_*C_{n+1}X$, $n > 0$, together with a catalogue of the relations amongst the operations.

In more detail, Section 1 gives a list of the commutation relations between all of the operations, coproduct, product, and between them and the Steenrod operations, conjugation, and homology suspension. The relationship between Whitehead products and the λ_n is also described.

Section 2 contains the definition of certain algebraic structures naturally suggested by the preceding section; the free versions of these algebraic structures are constructed.

Figure III.6.0.2. Excerpt from Cohen, Frederick R.; Lada, Thomas J.; May, J. Peter. The homology of iterated loop spaces. Lecture Notes in Mathematics. 533. Berlin-Heidelberg-New York: Springer-Verlag. VII, 490 p. (1976).

III.7. Factorization homology (not covered in spoken lecture)

It turns out that the $\mathbb{E}_n$ operads allow for definitions that one can articulate purely using smooth geometry of manifolds – see Exercise III.22. One then understands that an $\mathbb{E}_n$ algebra A encodes data compatible with various ways one can embed n-dimensional disks inside a smooth manifold called $\mathbb{R}^n$.

It is only natural to try to see if each such an algebra, then, has enough data to understand how disks embed in other smooth n-dimensional manifolds.[17]

It turns out that by "integrating[18] out" all such ways of embeddings disks into a given manifold M, one obtains an invariant called the *factorization homology* of M with coefficients in A. Because you have integrated over the entire manifold, the resulting object has an action by all the diffeomorphisms of M.[19]

For example, let A be an $\mathbb{E}_1$-algebra (your favorite example may be an associative algebra, or an A_∞-algebra). There aren't many 1-manifolds to consider. It turns out that the factorization homology

$$\int_{S^1} A$$

of the circle with coefficients in A is precisely the "Hochschild homology complex" of A.[20] What's more, the previous paragraph should convince you that there is a natural circle action on Hochschild chains. If you were to use a purely "chain-level" or algebraic model for everything, this circle action takes some work to write down; but it is almost trivially manifest when we set up this theory as in the previous paragraphs. See works of Francis, Ayala-Francis, Lurie, and Ayala-Francis-Tanaka.

By now the theory has exploded to incorporate stratified manifolds of various types, and promises to give a manifold-style framework for (∞, n)-categories.[21] It is typically a huge amount of painful work to fit geometric structures into higher-categorical frameworks, so successes in this direction are hugely desirable for many topologists. (If two of our main tools are smooth manifolds and higher category theory, we would love to have an efficient bridge to utilize both at the same time.) For example, such a

[17] Technically one must frame the manifold, but if one wants to understand manifolds that aren't framed, one could instead demand our $\mathbb{E}_n$ algebras have more structures; for example, $O(n)$-actions compatible with the $\mathbb{E}_n$ algebra structure.

[18] That is, by taking an appropriate colimit indexed by the diagram of all possible embeddings. This can also be phrased conveniently as a left Kan extension out of a particular category of smooth manifolds.

[19] Or, framed diffeomorphisms.

[20] If A is an $\mathbb{E}_1$-algebra in chain complexes, this is true on the nose. There are more general definitions of Hochschild homology that apply beyond the setting of chain complexes, including in the setting of spectra.

[21] See David Ayala, John Francis, Nick Rozenblyum, "A stratified homotopy hypothesis." arXiv:1502.01713v4.

framework may help establish deep theorems linking smooth topology to higher algebra – a clear target at the moment is to attain a proof of the cobordism hypothesis of Baez-Dolan, Lurie, and Hopkins-Lurie.

Exercises about planar operads

These exercises deals with the definition of *planar operad*, which are informally operads without symmetric group actions. Many features of operads become to fix

III.7.1. Trees (for the sake of visualizing the associative condition). To organize the combinatorics, let $\underline{n} = \{1 < \ldots < n\}$ denote the linearly ordered set with n elements. Every surjection $p : \underline{n} \to \underline{m}$ gives a partition of $\underline{n}$ into $\underline{m}$ subsets. When discussing planar operads, we will also only consider those p that respect order (so $i \leq j \implies p(i) \leq p(j)$).

REMARK III.7.1.1. I encourage you to imagine $\underline{m}$ as a set of leaves of a tree with one internal vertex (I will call this a corolla – a tree with only one non-leaf, non-root vertex). The surjection p should be imagined as encoding the data of a n-leafed corolla being split apart, then grafted onto the m-leafed corolla. Equivalently, p yields the data of m disjoint trees, where the ith tree has $p^{-1}(i)$ leaves; by grafting the ith tree onto the ith leaf of the corolla with m leaves, one obtains a tree with n leaves; this tree is not a corolla.

REMARK III.7.1.2. Now imagine you have a composition of surjections $\underline{n} \xrightarrow{p} \underline{m} \xrightarrow{q} \underline{l}$. The process of the previous remark results in a tree with n leaves, but there are two equivalent ways in which one could have created this large tree. (I) By first creating m disjoint trees with the kth tree having $p^{-1}(k)$ many leaves, as prescribed by p, then grafting the m trees together by q; or (II) by first grafting trees as prescribed by q, then for each leaf $i \in \underline{m}$, grafting trees as prescribed by p. These give rise to the two maps being compared in the associativity condition (i) below.

REMARK III.7.1.3. Now consider a function $p : \underline{n} \to \underline{m}$ that need not be surjective. When $p^{-1}(i)$ is empty, one should imagine that the ith leaf is grafted to an "empty tree," and as though the whole branch with the ith leaf disappears. The end result of the grafting is still a tree with n leaves.

III.7.2. Definition and examples.

DEFINITION III.7.2.1. A *planar operad* $\mathcal{O}$ (in sets) is the data of:

(a) (Sets of k-ary operations.) Sets $\mathcal{O}(k)$ for every integer $k \geq 0$, and

(b) (Composition maps.) For every (not necessarily surjective) order-respecting function $p : \underline{j} \to \underline{k}$, letting $j_i = \#(p^{-1}(i))$, we have functions

$$\gamma_p : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \mathcal{O}(j_2) \times \ldots \times \mathcal{O}(j_k) \to \mathcal{O}(j).$$

We demand these satisfy the following associativity condition:

(i) For functions $\underline{j} \xrightarrow{p} \underline{k} \xrightarrow{q} \underline{l}$, let us write p as a disjoint union of functions $p = \coprod_{i=1}^{l} p_i$ where $p_i = p|_{(qp)^{-1}(i)}$. We demand

$$\gamma_{qp}(id_{\mathcal{O}(l)} \times \prod_{i=1}^{l} \gamma_{p_i}) = \gamma_p(\gamma_q \times \mathrm{id}).$$

Remark III.7.2.2. Let $\mathcal{V}$ be any monoidal category. (For example, vector spaces with tensor product, which is even symmetric monoidal.) Then it makes sense to speak of planar operads in $\mathcal{V}$, where each $\mathcal{O}(k)$ is an object of $\mathcal{V}$ and we replace every instance of direct product with the monoidal product.

Common instances include: the category of vector spaces with tensor product over the base field, and the category of topological spaces with direct product. Of course, these examples are *symmetric* monoidal. Here is an example of a non-symmetric monoidal category: Fix an associative, non-commutative ring R, and consider the category of R-bimodules. The monoidal structure $\otimes_R$ is not symmetric monoidal.

III.8. The 1-ary space has an associative product

Given a planar operad $\mathcal{O}$, show that $\mathcal{O}(1)$ is endowed with a (possibly non-unital) associative binary operation.

Hint: When $p = \mathrm{id}_{\underline{1}}$, γ_p defines the multiplication and (i) states the multiplication is associative.

Example III.8.0.1. If $\mathcal{V}$ is the category of vector spaces over k, then $\mathcal{O}(1)$ is some k-linear, possibly non-unital associative ring.

Remark III.8.0.2 (The 0-ary operations.). It is common to assume that $\mathcal{O}(0)$ is the monoidal unit – so for an operad in sets, to assume $\mathcal{O}(0)$ is a point, and for an operad in vector spaces, to assume $\mathcal{O}(0)$ is the base field.

In fact, it is also common to change the definition of "planar operad" to not even require a 0-ary space of operations.

In all examples we see in this course, we will have that $\mathcal{O}(0)$ is the monoidal unit, and that the composition maps $\gamma_{\emptyset \to \emptyset}$ are the natural isomorphism $\mathcal{O}(0) \otimes 1^{\otimes} \to \mathcal{O}(0)$ (where $1^{\otimes}$ is the monoidal unit).

III.9. Examples of planar operads

All the following examples can in fact be made into operads (not just planar operads) but this exercise is meant to get you used to the non-equivariance-related ingredients of an operad.

(a) Let $\mathcal{O}(k) = \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^{\coprod k}, \mathbb{R})$ be the collection of orientation-preserving smooth, open embeddings of k disjoint copies of $\mathbb{R}$ into $\mathbb{R}$. Endow the collection $\mathcal{O}(k), k \geq 0$ with the structure of a planar operad. (In fact, one can make this an operad.) Convince yourself it is a planar operad in spaces – i.e., that your composition maps are continuous.

(b) Let $\mathcal{V}$ be the category of vector spaces and fix a vector space V. Then we have the *endomorphism planar operad*, often denoted End. This has k-ary operations given by

$$\mathrm{End}(k) := \hom(V^{\otimes k}, V).$$

The composition map takes a tuple

$$f \otimes g_1 \otimes \otimes g_k \in \hom(V^{\otimes k}, V) \otimes \hom(V^{\otimes j_1}, V) \otimes \ldots \otimes \hom(V^{\otimes j_k}, V)$$

and outputs the composition

$$f \circ (g_1 \otimes \ldots \otimes g_k).$$

Show that this is a planar operad in vector spaces. (This planar operad may in fact be promoted to an operad.)

(c) The endomorphism planar operad makes sense in any (not-necessarily-symmetric) monoidal category. In fact, show that the first example of this exercise is a planar endomorphism operad for some monoidal category.

III.10. Unitality

There are secretly two ways in which operads can be unital. One has to do with $\mathcal{O}(0)$, and the other has to do with specifying a distinguished element in the space of 1-ary operations. This exercise deals with the latter.

Definition III.10.0.1. We say that a planar operad $\mathcal{O}$ in sets is *unital* if

(a) there exists an element $1 \in \mathcal{O}(1)$ satisfying the following conditions:
- For the map $p : \underline{j} \to \underline{1}$,
$$\gamma_p(1, -) = \mathrm{id}_{\mathcal{O}(j)}\,.$$
- For $p = \mathrm{id} : \underline{j} \to \underline{j}$,
$$\gamma_{\mathrm{id}}(-, 1, 1, \ldots, 1) = \mathrm{id}_{\mathcal{O}(j)}\,.$$

More generally, a planar operad in a monoidal category $\mathcal{V}$ is unital if there exists a map

$$u : 1^{\otimes} \to \mathcal{O}(1)$$

(where the domain is the monoidal unit of $\mathcal{V}$) for which

$$\gamma_p \circ (u \otimes \mathrm{id}_{\mathcal{O}(j)}) = \mathrm{id}_{\mathcal{O}(j)} \qquad \text{and} \qquad \gamma_{\mathrm{id}_{\underline{j}}} \circ (\mathrm{id}_{\mathcal{O}(j)} \otimes u \otimes \ldots \otimes u) = \mathrm{id}_{\mathcal{O}(j)}\,.$$

(a) Show that if a planar operad is unital, then the unit $1 \in \mathcal{O}(1)$ is unique. This follows from Exercise III.8 and the usual proof that units of monoids are unique.

(b) In our examples above, show that there is the identity map $\mathrm{id} \in \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R})$ and $\mathrm{id} \in \hom(V, V)$.

III.11. Maps of planar operads

DEFINITION III.11.0.1. Let $\mathcal{O}$ and $\mathcal{P}$ be planar operads in sets. A *map of planar operads* from $\mathcal{O}$ to $\mathcal{P}$ is the data of maps

$$f_k : \mathcal{O}(k) \to \mathcal{P}(k), \qquad k \geq 0$$

so that for every function $p : \underline{j} \to \underline{k}$, we have

$$f_{j_1+\ldots+j_k}(\gamma_p) = \gamma_p \circ (f_k \times f_{j_1} \times \ldots \times f_{j_k}).$$

f is further a *map of unital planar operads* if $f_1(1) = 1$.

(a) Show there is a terminal planar operad in sets for which $\mathcal{O}(k) = *$ for all k. This is the *planar associative operad*. (For $\mathcal{O}$ to be terminal means that, for any other planar operad $\mathcal{O}'$, there exists a unique map of planar operads from $\mathcal{O}'$ to $\mathcal{O}$.)
(b) Show the planar associative operad is terminal with or without the demand for unitality.
(c) Suppose $\mathcal{O}$ is the planar associative operad, and let End_M be the endomorphism planar operad of some set M. Show that a map of unital planar operads $\mathcal{O} \to \mathrm{End}_M$ precisely encodes an associative algebra structure on M. This explains why we call the terminal planar operad associative.

REMARK III.11.0.2. Let R be a ring and M an abelian group. Recall that the data of a left R-module structure on M is equivalent to the data of a ring homomorphism

$$R \to \mathrm{End}(M)$$

to the set of abelian group endomorphisms. Note that, to utilize this definition of a module, we need to know what the ring structure on $\mathrm{End}(M)$ is.

Exercise III.11 witnesses an analogous situation for operads. Given a planar operad $\mathcal{O}$, it seems that a natural notion of an "$\mathcal{O}$-algebra" is encoded by a map of planar operads

$$\mathcal{O} \to \mathrm{End}_M .$$

However, a diligent exploration shows that there is not enough structure in the planar endomorphism operad to ever know whether a map from some operad $\mathcal{P}$ detects a *commutative* algebra structure on M. This motivates the notion of a *symmetric* operad, or operad for short.

Exercises on (Symmetric) operads

We have seen that planar operads have the potential to encode algebraic structures, but failed to encode commutative structures in a natural way. This was probably recognized quite early on – precursors to Peter May's operads (including Adams and Mac Lane's PROPs[22]) all incorporated symmetric group actions.

To emphasize the symmetric group actions, what we are about to define is often called a "symmetric operad." But we view this notion as the basic notion, so we simply call the following an operad, just as May did in his original work[23]. Of course one should compare this with Definition III.7.2.1.

Definition III.11.0.3 (Operads). A *symmetric operad in sets*, or operad in sets, is the data of

(a) For each $n \geq 0$, a set $\mathcal{O}(n)$ with a right action by the symmetric group Σ_n on $\mathcal{O}(n)$, and
(b) For every (not necessarily order-respecting, not necessarily surjective) function $p : \underline{j} \to \underline{k}$, a composition map

$$\gamma_p : \mathcal{O}(k) \times \prod_{i=1}^{k} \mathcal{O}(j_i) \to \mathcal{O}(j).$$

We demand

(i) that these satisfy the same associativity condition as in Definition III.7.2.1 (i), and
(ii) that each γ_p is equivariant in the following sense: For every $\sigma \in \Sigma_k$, we have an equality

$$\tilde{\sigma} \circ \gamma_{\sigma \circ p} \circ (\mathrm{id}_{\mathcal{O}(k)} \times \sigma_{\prod_{i=1}^{k} \mathcal{O}(j_i)}) = \gamma_p \circ (\sigma \times \mathrm{id}_{\prod_{i=1}^{k} \mathcal{O}(j_i)}). \tag{III.11.1}$$

Further, for any tuple of $\tau_i \in \Sigma_{j_i}$, we demand that

$$\gamma_p(\mathrm{id}_{\mathcal{O}(k)} \times \prod_{i=1}^{k} \tau_i) = (\prod_{i=1}^{k} \tau_i) \circ \gamma_p. \tag{III.11.2}$$

[22] Saunders MacLane, "Categorical algebra," Bull. Amer. Math. Soc. 71 (1965), 40-106. DOI `https://doi.org/10.1090/S0002-9904-1965-11234-4`. See Chapter V.

[23] J. P. May, The Geometry of Iterated Loop Spaces, Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg 1972. A free and TeXed version is available on Peter's website.

On the righthand side, $\prod_{i=1}^k \tau_i$ denotes the obvious element of Σ_j.

Notation III.11.0.4. Let us explain the notation in Condition (ii).

First, $\tilde{\sigma}$ is a bijection of $\underline{j}$ to itself; we define it as the unique bijection satisfying $p \circ \tilde{\sigma} = \sigma \circ p$ and for which the restriction of $\tilde{\sigma}$ to a fiber $p^{-1}(i)$ is order-preserving as a map to $\underline{j}$. In the equality (III.11.1), $\tilde{\sigma}$ (by abuse of notation) is also the map $\mathcal{O}(j) \to \mathcal{O}(j)$ given by the action of Σ_j supplied by (a).

Second, $\sigma_{\prod_{i=1}^k \mathcal{O}(j_i)}$ is the isomorphism

$$\prod_{i=1}^k \mathcal{O}(j_i) \xrightarrow{\cong} \mathcal{O}(j_1)\times\ldots\times\mathcal{O}(j_k) \xrightarrow{\cong} \mathcal{O}(j_{\sigma^{-1}(1)})\times\ldots\times\mathcal{O}(j_{\sigma^{-1}(k)}) \xrightarrow{\cong} \prod_{i=1}^k \mathcal{O}(j_{\sigma^{-1}(i)})$$

supplied by the symmetric monoidal structure of direct product with sets.

Finally, we note that the abuse of notation – denoting σ for both a symmetric group element and the induced automorphism of $\mathcal{O}(k)$ – may result in some confusion in (III.11.2), which we now seek to undo. On the righthand side, $\prod_{i=1}^k \tau_i$ is an element of Σ_j, while on the lefthand side, one has the map

$$\mathcal{O}(j_1) \times \ldots \times \mathcal{O}(j_k) \xrightarrow{\tau_1\times\ldots\times\tau_k} \mathcal{O}(j_1) \times \ldots \times \mathcal{O}(j_k).$$

Before we go any further, let us put for the record:

Definition III.11.0.5. Let $\mathcal{O}$ and $\mathcal{P}$ be operads. Then a *map of operads* is the data, for every $k \geq 0$, of a function $f_k : \mathcal{O}(k) \to \mathcal{P}(k)$ that respects composition, and which is Σ_k-equivariant.

Remark III.11.0.6 (Unitality). By imposing the same conditions as in the planar case, one obtains definitions of *unital* operad, and map of unital operads.

Definition III.11.0.7. An (unital) *algebra* over an operad $\mathcal{O}$ is the data of an object V together with a unital map of operads $\mathcal{O} \to \mathcal{E}\mathrm{nd}_V$.

Given such data, we say V is an $\mathcal{O}$-algebra.

III.12. Symmetric sequences

Let Σ denote the category whose objects are finite sets and whose morphisms are bijections. There is of course an equivalence $\Sigma \cong \Sigma^{\mathrm{op}}$, but we will take care and declare a *symmetric sequence* in a category $\mathcal{C}$ to be a functor

$$\Sigma^{\mathrm{op}} \to \mathcal{C},$$

meaning a symmetric sequence is determined by the data of objects $X(n)$ for all $n \geq 0$, equipped with a *right* Σ_0 action. We let $\mathcal{C}^{\Sigma}$ denote the category of symmetric sequences (where morphisms are natural transformations). If

$\mathcal{C}$ has colimits and a symmetric monoidal structure $\otimes$ preserving colimits in each variable, one has a *not symmetric* monoidal structure $\circ$ on $\mathcal{C}^\Sigma$, where

$$(X \circ Y)(n) = \bigoplus_{k \geq 1} \bigoplus_{n_1 + \ldots + n_k = n} X(k) \otimes_{\Sigma_k} (Y(n_1) \otimes \ldots \otimes Y(n_k)) \otimes_{\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}} k[\Sigma_n].$$

(a) Show that an operad in $\mathcal{C}$ is the same thing as an associative monoid in $\mathcal{C}^\Sigma$ (with respect to the composition product $\circ$).

III.13. From planar to symmetric

There is a forgetful functor from operads to planar operads, which forgets the symmetric group actions. Show that there is a left adjoint. Or, just convince yourself that there is a natural way to upgrade any planar operad to a symmetric operad (do not just give things the trivial Σ_k actions).

III.14. Examples of operads: Endomorphism operad

Let $\mathcal{V}$ be a symmetric monoidal category, and fix an object V. We let $\mathcal{E}\mathrm{nd}_V$ denote the operad where the k-ary operation space is given by

$$\mathcal{E}\mathrm{nd}_V(k) := \hom(V^{\otimes k}, V)$$

with the obvious right symmetric group action. (Given $\sigma \in \Sigma_k$, we use the symmetric monoidal structure of $\mathcal{V}$ to permute the factors $V^{\otimes k} \cong V^{\otimes k}$. This defines a left action on $V^{\otimes k}$, hence a right action on $\hom(V^{\otimes k}, W)$ for any target W.) The composition maps are, as in the planar operad case, composition maps.

(a) Elucidate the equivariance condition in this setting to show that $\mathcal{E}\mathrm{nd}_V$ is a (symmetric) operad when $\mathcal{V}$ is the category of sets with direct product, and V is just a set.

For concreteness, if you want, you can just take $\mathcal{V}$ to be the category of sets with direct product and following along the work here: Fix

$$g : V^{\times k} \to V, \qquad f_i : V^{\times j_i} \to V, i = 1, \ldots, k.$$

Then both sides of first equivariance condition (III.11.1), applied to the tuple $(g, f_1, \ldots, f_k)$, equal

$$g \circ (f_{\sigma^{-1}(1)} \times \ldots \times f_{\sigma^{-1}(k)}).$$

Let us work out the lefthand side of (III.11.1) for the reader's convenience:

$$\begin{aligned}
&\left(\tilde{\sigma} \circ \gamma_{\sigma \circ p} \circ (\mathrm{id}_{\mathcal{O}(k)} \times \sigma_{\prod_{i=1}^k \mathcal{O}(j_i)})\right)(g, f_1, \ldots, f_k) \\
&= (\tilde{\sigma} \circ \gamma_{\sigma \circ p})\,(g, f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(k)}) \\
&= \tilde{\sigma}(g \circ (f_1 \times \ldots \times f_k)) \\
&= g \circ (f_{\sigma^{-1}(1)} \times \ldots \times f_{\sigma^{-1}(k)})
\end{aligned}$$

(b) Recall from Example III.2.0.6 that the endomorphism operad for a chain complex is an operad (in chain complexes). Elucidate the equivariance condition in this setting when $\mathcal{V}$ is the category of chain complexes over a field k, and $\otimes_k$ is the symmetric monoidal structure. I am very sorry to say that I want you to be conscious of signs. You will also need to dust off the definition of the hom-cochain complex.

III.15. Examples of operads: The associative operad

For every integer $k \geq 0$, fix a set of cardinality k. We let $\mathcal{A}\mathrm{ss}$ be the operad whose k-ary space is the set of all possible linear orderings on the set of cardinality k. (This is a set with a free Σ_k action.) Composition

$$\mathcal{A}\mathrm{ss}(k) \times \mathcal{A}\mathrm{ss}(j_1) \times \ldots \times \mathcal{A}\mathrm{ss}(j_k) \to \mathcal{A}\mathrm{ss}(j_1 + \ldots + j_k)$$

is given by inducing the lexicographic ordering.

One should interpret the freeness of the symmetric actions as saying: The operations that a map out of $\mathcal{A}\mathrm{ss}$ picks out are completely determined by knowing what one element of $\mathcal{A}\mathrm{ss}(k)$ picks out (by using the equivariance condition), with no conditions on how the operation behaves when inputs are permuted.

(a) Letting $\mathcal{E}\mathrm{nd}_V$ be the endomorphism operad of a set V, show that a unital map of operads $\mathcal{A}\mathrm{ss} \to \mathcal{E}\mathrm{nd}_V$ precisely specifies an associative monoid structure on V.

(b) Formulate what the "associative operad in vector spaces" should be, with the goal that your answer should classify (unital) associative algebra structures. (That is, for any vector space V a map of operads from your answer to $\mathcal{E}\mathrm{nd}_V$ should precisely endow V with a unital associative algebra structure.)

(Hint: The k-ary space should be the free vector space generated by the set $\mathcal{A}\mathrm{ss}(k)$. A non-canonical way to think about this vector space is as the group ring on the symmetric group on k letters.)

III.16. Examples of operads: The commutative operad

We let $\mathcal{C}\mathrm{omm}$ be the operad whose k-ary spaces are all a single point, endowed with the trivial Σ_k action. We caution the reader that this has the same "underlying k-ary operation space" as the planar associative operad, but it is most definitely not the(symmetric) associative operad.

(a) Show that $\mathcal{C}\mathrm{omm}$ is the terminal operad, meaning any other operad has a unique operad map to $\mathcal{C}\mathrm{omm}$.

(b) Letting $\mathcal{E}\mathrm{nd}_V$ be the endomorphism operad of a set V, show that a unital map of operads $\mathcal{C}\mathrm{omm} \to \mathcal{E}\mathrm{nd}_V$ precisely specifies a commutative monoid structure on V.

(c) Conclude that for any operad in sets $\mathcal{O}$, and for any commutative unital monoid V, V has a canonical $\mathcal{O}$-algebra structure. (As a sanity check, you should consider the case that $\mathcal{O}$ is the associative operad.)

(d) Formulate what the "commutative operad in vector spaces" should be, with the goal that your answer should classify (unital) commutative algebra structures.
(e) Is the commutative operad in vector spaces the terminal operad (in vector spaces)?

III.17. Non-examples

There are many kinds of algebraic structures that have no operad. So for example, the theory of operads is not at all equivalent to the theory of "model theory" that logicians like.

(a) Show that there is no operad in sets whose algebras are groups.
(b) Show that there is no operad in abelian groups whose algebras are fields.

III.18. When the tensor is a coproduct

Suppose that $\mathcal{C}^{\otimes}$ is a symmetric monoidal category where $\otimes$ is given by *coproduct*. An example would be $\mathcal{C} = \mathsf{Ab}$ the category of abelian groups with direct sum as the monoidal structure, or $\mathcal{C} = \mathsf{Sets}$ the category of sets with disjoint union.

(a) Show that for any object $V \in \mathcal{C}$, there is a unique map from the commutative operad to the endomorphism operad of V.

III.19. Versions of operads in sets

Things like vector spaces and abelian groups have "underlying sets," which are the images under the forgetul functor to Sets. As you have seen, we have a left adjoint to forgetting called "the free construction," which takes a set to a free vector space, or to a free abelian group. And in all linear settings, the free construction further can be made symmetric monoidal, so that it takes direct products of sets to the relevant $\otimes$ on your linear gadgets (e.g., $\otimes_{\mathbb{Z}}$ for abelian groups, or $\otimes_k$ for vector fields with base field k). For concreteness, we'll choose a base field k in this exercise.

(a) Suppose $\mathcal{C}^{\otimes}$ the category of k-vector spaces with symmetric monoidal structure given by $\otimes_k$. Convince yourself that any operad in $\mathcal{C}^{\otimes}$ in fact defines an operad in sets, by forgetting the linear structure. In fact, there is a forgetful functor from operads in $\mathcal{C}^{\otimes}$ to operads in sets.
(b) Using the free functor $\mathsf{Sets} \to \mathcal{C}$, construct (or argue there exists) a left adjoint to the forgetful functor from the previous part. In case it helps, for an operad $\mathcal{O}$ in sets, the induced operad in vector spaces should have j-ary operation space given by the k-vector space generated by the set $\mathcal{O}(j)$.

    The most important part of this exercise is to understand that if $\mathcal{O}$ is an operad in sets, and $\mathcal{P}$ is an operad in $\mathcal{C}^{\otimes}$, then a map of operads of sets

$$\mathcal{O} \to \mathcal{P}$$

precisely gives a map of operads in $\mathcal{C}^{\otimes}$ with target $\mathcal{P}$, and with domain the operad induced by $\mathcal{O}$.

(c) When $\mathcal{C}^{\otimes}$ is the category of k-vector spaces with tensor product $\otimes_k$, show that the "induced operad" generated by $\mathcal{C}$omm is the commutative operad in vector spaces that you made earlier. Or, you can do this for $\mathcal{A}$ss instead.

(d) When $\mathcal{P} = \mathcal{E}\mathrm{nd}_V$ for some vector space V, consider $\mathcal{P}$ as an operad in sets. (So $\mathcal{P}(j)$ is the *set* of linear homomorphisms $V \otimes \ldots \otimes V \to V$.) Convince yourself that a map of operads (in sets) from $\mathcal{C}$omm to $\mathcal{P}$ indeed gives a (unital) commutative algebra structure on V.

(e) In general, if $\mathcal{C}^{\otimes}$ is just a symmetric monoidal category, its hom sets are just sets. So for any object V, the endomorphism operad is an operad in sets.

Remark III.19.0.1. In instances where we are talking about operads in chain complexes, or in spectra, we still have a forgetful functor to spaces.[24] The same yoga as in this exercise works to take any operad in spaces to produce an operad in chain complexes, or in spectra.[25] And in particular, to give a spectrum V a commutative algebra structure, it will suffice to give a "map of operads" from the space-level commutative operad to the endomorphism operad of the spectrum. I utilize quotes because I'm being coy about what the correct notion of what a map of operads ought to be.

Remark III.19.0.2. What's interesting is that there are operads that *do not arise* as being induced from operads in sets. An example is the Lie operad $\mathcal{L}$ie for vector spaces, where an algebra over $\mathcal{L}$ie is exactly a Lie algebra.

III.20. Maps of algebras

Fix an operad $\mathcal{O}$ in spaces, and two $\mathcal{O}$-algebras A and B (in spaces, for concreteness). We did not discuss in the lectures what a map of $\mathcal{O}$-algebras from A to B is.

(a) Formulate what you believe to be a reasonable definition. You can be as unsophisticated as you like – for example, you can say that a map of algebras is a single continuous map $f : A \to B$ satisfying some properties. What properties should it satisfy?

I wouldn't recommend getting too much more sophisticated than this. If you've heeded the warnings from lecture, you know that this

[24] If you haven't seen the construction for chain complexes, let me at least give you something to look up: The Dold-Kan correspondence. This takes a chain complex to a simplicial Abelian group, and the underlying simplicial set (forgetting the abelian group structure) defines a space.

[25] We saw already how we can take $C_*\mathcal{O}(j)$ of any topological operad $\mathcal{O}$; for an operad in spectra, we can take $\Sigma^\infty(\mathcal{O}(j)+)$.

unsophisticated definition has to be wrong somehow to handle all homotopically interesting phenomena. But that's okay, because things will work out for the purposes of this exercise.

(b) Let $f : X \to Y$ be a continuous map. Show that $\Omega f : \Omega X \to \Omega Y$ is a map of $\mathbb{E}_1$-algebras.

(c) Recall that there is a map $\mathbb{E}_1 \to \mathbb{E}_2$ of operads. On the other hand, suppose you have a map $f : X \to \Omega Y$ and you consider the induced map $\Omega f : \Omega X \to \Omega^2 Y$. We have the intuition that Ωf ought to be a map of $\mathbb{E}_1$ groups, but $\Omega^2 Y$ is an $\mathbb{E}_2$-group. How does this play with the map $\mathbb{E}_1 \to \mathbb{E}_2$?

Warning III.20.0.1. I promise it was a good exercise to try to define what a map of $\mathcal{O}$-algebras is. But let me warn you that (for example) using your definition[26], if you take the example of $\mathcal{O} = A_\infty$ as an operad in chain complexes, a map of A_∞-algebras would not be the usual definition of the map of A_∞-algebras.

[26]Unless you knew some tricks like utilizing Koszul duality

Exercises on $\mathbb{E}_1, \mathbb{E}_n, \mathbb{E}_\infty$

III.21. $\mathbb{E}_n$ using configuration spaces

(a) For every $k, n \geq 0$, show that $\mathbb{E}_n(k)$ is homotopy equivalent to the configuration space of k ordered points in $\mathbb{R}^n$.
(b) Show that $\mathbb{E}_n(k)/\Sigma_k$ is homotopy equivalent to the configuration space of k unordered points in $\mathbb{R}^n$.
(c) Explain why the homology of $\mathbb{E}_n(k)/\Sigma_k$ computes the Σ_k-equivariant homology of $\mathbb{E}_n(k)$ correctly (e.g., as though you used the Borel construction).
(d) In the case $n = \infty$, which well-known algebraic invariant of the group Σ_k does the homology of $\mathbb{E}_\infty(k)/\Sigma_k$ compute?

III.22. $\mathbb{E}_n$ using smooth manifolds

Let X be a smooth manifold. A *framing* of X is a trivialization of the tangent bundle.

Let X and Y be two smooth, framed manifolds of the same dimension. A *framed embedding* from X to Y is the data of (i) A C^∞ embedding $j : X \to Y$, and (ii) a homotopy of the framing of X to the pulled-back framing induced by j.[27]

(a) Fix the standard framing on $\mathbb{R}^n$. For every $k, n \geq 0$, convince yourself that $\mathbb{E}_n(k)$ is homotopy equivalent to the space of framed embeddings of $\coprod_k \mathbb{R}^n$ into $\mathbb{R}^n$.

III.23. The $\mathbb{E}_\infty$ operad

(a) Prove that for every $k \geq 0$, $\mathbb{E}_\infty(k)$ is contractible. (Hint: Forgetting a cube gives a continuous map from $\mathbb{E}_\infty(k)$ to $\mathbb{E}_\infty(k-1)$. The hard part is showing that this is a fibration, which you might ignore for sake of

[27] Homotopy-theoretically, it is healthy to think of a framing of X as a homotopy-nullification of the map $X \to BO(n)$ classifying the tangent bundle of X. (Here, $n = \dim X$.) In other words, a framing is a homotopy making the diagram

$$\begin{tikzcd} & * \arrow[d] \\ X \arrow[ur] \arrow[r] & BO(n) \end{tikzcd}$$

commute. Then a framed embedding also carries the data of a homotopy between the nullification of X, and the nullification induced by j.

expediency. Now try to think through why the fibers have vanishing homotopy groups.)

(b) Show that the $\mathbb{E}_\infty$ operad is equivalent to $\mathcal{C}\mathrm{omm}$ (as operads in spaces).

III.24. Suspension spectra give rise to $\mathbb{E}_\infty$-algebras

(a) Let X be a pointed topological space. Using the model of $\Omega^\infty\Sigma^\infty X$ (also known as QX) as an honest increasing union, explain why QX is an algebra over the E_∞-operad. (Hint: The map $\Omega\Sigma X \to \Omega^2\Sigma^2 X$ is modeled as Ω of the map $\Sigma X \to \Omega\Sigma(\Sigma X)$. It may also help to look at later parts of Exercise III.20.)

(b) Let Y be any spectrum. Try to give Y_0 the structure of an $\mathbb{E}_\infty$ algebra. What is an additional complication that distinguishes your efforts from the previous problem?

(c) For some time, Peter May considered sequences of spaces Y_i together with *homeomorphisms* $Y_i \xrightarrow{\cong} \Omega Y_{i+1}$. (Such spectra are sometimes called May spectra[28].) Why does this help with the issue you encountered in the previous problem?

Remark III.24.0.1. From this exercise, you can see how suspension spectra, and May spectra, could have felt so right. Our notions of composition (expressed through commutativity of diagrams) demanded equalities (in our definitions of operads, and of algebras over operads). This strictness handcuffs us.

Being an E_∞-algebra in spaces is very different from being a commutative algebra in spaces. For example, QS^0 is an E_∞-algebra by the above exercise, but it cannot be made into a topological abelian group (Exercise II.20).

III.25. Stasheff polytopes (a very particular model of the planar A_∞-operad)

In a healthy mathematical universe, both the associative and A_∞-operads should be considered manifestly equivalent. This is because both operads have n-ary operations spaces on which Σ_n acts freely and transitively on connected components, and where each connected component is contractible. More concretely, any operad called an A_∞ operad should have an evident map to $\mathcal{A}\mathrm{ss}$ exhibiting an equivalence. It is an artifact of history that $\mathcal{A}\mathrm{ss}$ and A_∞ are treated differently (and in fact, many modern homotopy-theorist will dispose of any distinction). I think it's fair to say that, historically, the

[28]But for us, we might as well treat this as a non-detectable condition. We want to embrace a philosophy where the only equivalence is a homotopy equivalence, and in particular, we should recognize that the notion of homeomorphism is a notion particular to the model of spaces, and not articulable in any truly homotopical model of homotopy theory.

two operads were treated differently only because the correct notion of the space of "maps of operads" was very slow to develop.

Regardless, some models of the A_∞ operad are surprisingly effective and convenient. Here we'll explore a little of the Stasheff associahedron model, to see how the algebraic A_∞-relations (which are themselves consequences of an arbitrary/particular model for the A_∞-operad) are derived from the Stasheff associahedron model.

Definition III.25.0.1 (Corollas). For us, a *corolla* is a planar rooted tree whose only vertices are leaves and a root. So for every $k \geq 1$, there is exactly one corolla with k leaves. The $k = 1$ corolla is just a vertex; the $k = 2$ corolla is a planar rooted tree with three vertices, two of which point to the third; the $k = 3$ corolla is a directed planar tree whose three leaves point to the root; and so forth.

Definition III.25.0.2 (Stasheff polyhedra, informally). We will define the kth Stasheff associahedron inductively. For each $k \geq 0$, we define a topological space K_k called the kth associahedron, as follows:

For $k = 0, 1, 2$, the kth Stasheff associahedron is a point. You should think of the $k = 2$ case as saying that there is a unique binary planar rooted tree with no 1-ary (1-ary means one-input and one-output) vertices. This unique binary planar rooted tree is the corolla.

For each k (and I recommend you begin by following along for $k = 3$) one can construct a topological space called $\partial(K_k)$ as follows: Note that every planar rooted tree T with k leaves and with no 1-ary vertices, is either a corolla or a concatenation/grafting of corollas T_α. And for every corolla T_α making up T with k_α leaves, by induction we have already defined the k_α associahedron K_{k_α}. We define

$$\partial(K_k) := \bigcup_T \left(\prod_{T_\alpha} K_{k_\alpha} \right)$$

where the union runs over the collection of non-corolla planar rooted trees T with no 1-ary vertices and with k leaves; as for the direct product, for each corolla T_α in T with k_α leaves, we have a factor consisting of the k_αth Stasheff polyhedron. The union is not a disjoint union; there is an anodyne/harmless gluing procedure whose intuition will be come accessible as you work out some examples.

Finally, it turns out one can prove that $\partial(K_k)$ is homeomorphic to a sphere of dimension $(k-3)$. We define K_k to be the CW complex (homeomorphic to a disk) obtained by attaching a $(k-2)$-dimensional disk to this boundary sphere along a choice of homeomorphism.

Remark III.25.0.3. This is a non-ideal definition, in that induction is always difficult to work with, and the gluing procedure is annoying to write out. The above definition certainly makes any universal property completely inaccessible. Hiro's favorite definition of the associahedra, which isn't written up anywhere published, is as a nerve of a category of certain tree posets. If you ask Hiro he'd be happy to send you a draft of something describing this definition.

(a) Draw the associahedra for $k = 2, 3, 4$. You should get a point, a line interval, and a pentagon. (This is the same pentagon, philosophically, appearing in MacLane's pentagon for monoidal categories.)

(b) By virtue of the inductive definition, each K_k is a CW complex. Moreover, once you (arbitrarily) orient the $k = 3$ case, you can induce an orientation for all K_k. (Work out the $k = 4$ pentagon's orientation, paying attention to the boundary.)

Writing the degree $k-2$ generator of the top cell of K_k as m_k, write out the cellular chain complex for $k = 2, 3, 4$.

(c) Convince yourself that the K_k form a planar operad in spaces. Convince yourself that the cellular chains of K_k (after choosing orientations if your base ring R is not characteristic 2) form a planar operad in chain complexes.

(d) Suppose there is an operad in spaces whose k-ary space is given by K_k. Letting V be a cochain complex, write out what it means for you to have a map of chain complexes – from the cellular chains of K_k to $\hom(V^{\otimes_k k}, V)$ – for the case $k = 2, 3, 4$. Compare to the A_∞ relations for m_2, m_3, m_4.

(Some points of caution: Most people take *cohomological* grading in the A_∞ world, so it will behoove you to think of the cellular chain complexes concentrated in non-negative degrees as a cochain complex concentrated in non-positive degrees. And, if you are wondering where all the m^1 terms in the A_∞-relations show up, make sure to remember what the differential on a hom complex is.)

III.26. Equivalent models of the $\mathbb{E}_1$ operad, also known as the A_∞-operad

. For every one of the following operads, exhibit an equivalence of operads to $\mathcal{A}$ss.

(a) Let $\mathcal{O}(k) = \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^{\coprod k}, \mathbb{R})$ be the collection of orientation-preserving smooth, open embeddings of k disjoint copies of $\mathbb{R}$ into $\mathbb{R}$. Endow the collection $\mathcal{O}(k), k \geq 0$ with the structure of a planar operad. (In fact, one can make this an operad.) Convince yourself it is a planar operad in spaces – i.e., that your composition maps are continuous.

(b) The $\mathbb{E}_1$ operad.

(c) The operad defined using the free symmetric operad given by the Stasheff polyhedra as a planar operad. See Exercises III.25 and III.13

III.27. Lie Operad

Fix a base ring k. (You can use any base ring if you want, but be careful in characteristic 2.) Look up, or construct, the operad $\mathcal{L}$ie in R-modules – verify that your definition is correct by showing that a $\mathcal{L}$ie-algebra in R-modules is precisely a Lie algebra whose bracket is R-bilinear.

III.28. Poisson Operad

Fix a base ring k. (You can use any base ring if you want, but be careful in characteristic 2.)

Recall that a Poisson algebra over k is the data of:

(i) A k-module A,
(ii) A commutative unital algebra structure on A (where the product is k-bilinear), and
(iii) A Lie algebra structure $\{-,-\}$ on A (for which the bracket is k-bilinear),

such that For every $a \in A$, the bracket $[a,-]$ is a derivation for multiplication on A. That is,

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

Look up, or construct, the operad $\mathcal{P}$oisson in R-modules – verify that your definition is correct by showing that a $\mathcal{P}$oisson-algebra in R-modules is precisely a Poisson algebra over R.

Week 2

LECTURE IV

Ways operads show up in Floer theory: A_∞ relations

Where to begin? Operads show up everywhere in this game, and operads help us organize structures. In fact, I'll state a conjecture[1] very much motivated by some facts we can deduce from theorems about operads – so the organizational power of operads helps us motivate future directions. The two main examples I'd like to cover are the following: (i) How one model of the A_∞, or $\mathbb{E}_1$, operad shows up when studying holomorphic disks, and (ii) How a framed version of the $\mathbb{E}_2$ operad shows up when studying Riemann surfaces with k (marked) inputs and 1 (marked) output.

Convention IV.0.0.1. In this talk, every tree is planar (the leaves have a linear ordering), rooted (there is a specified root that is not a leaf – this directs the tree), and *stable*. Informally, stability means that a vertex of a tree is either a root, a leaf, or a point where at least 3 edges come together – that is, each non-leaf, non-root vertex has at least 2 incoming edges and exactly one outgoing edge.

Warning IV.0.0.2. Everything about this lecture concerns chain complexes. As a result, we have fixed a base ring R and each instance of $\otimes$ in this lecture is to be interpreted as $\otimes_R$. This is in violation of the conventions from Week One, when $\otimes$ always meant $\otimes_{\mathbb{S}}$ – i.e., the smash product of spectra.

Let me also remark that the tensor product of choice for complexes of R-modules should *always* be the derived tensor product $\otimes_R^{\mathbb{L}}$. But this will be immaterial for us – in this lecture, and in almost all set-ups of Floer theory, the chain complexes we consider will be generated by free R-modules, and hence flat.

IV.1. (One very popular model of) the A_∞ operad

First, let me tell you that A_∞ and $\mathbb{E}_1$ are synonyms. They describe exactly the same notion (of associativity up to higher and higher specific homotopy), and you should keep the example of ΩX as the main example

[1] As far as I know, it's still a conjecture

of an object with an A_∞-algebra structure. I want to emphasize that this isn't a theorem; it's a piece of vocabulary.[2]

But there is one model of the planar A_∞ operad that is so popular that this particular model has come to be called "the" A_∞ operad in many communities. This is the model utilizing what are commonly called the Stasheff associahedra. If you play a lot with chain complexes (like a lot of people using Fukaya categories), "the" A_∞-operad is the one you obtain by constructing a particular oriented cellular decomposition (and hence cellular chain complex) on the associahedra.

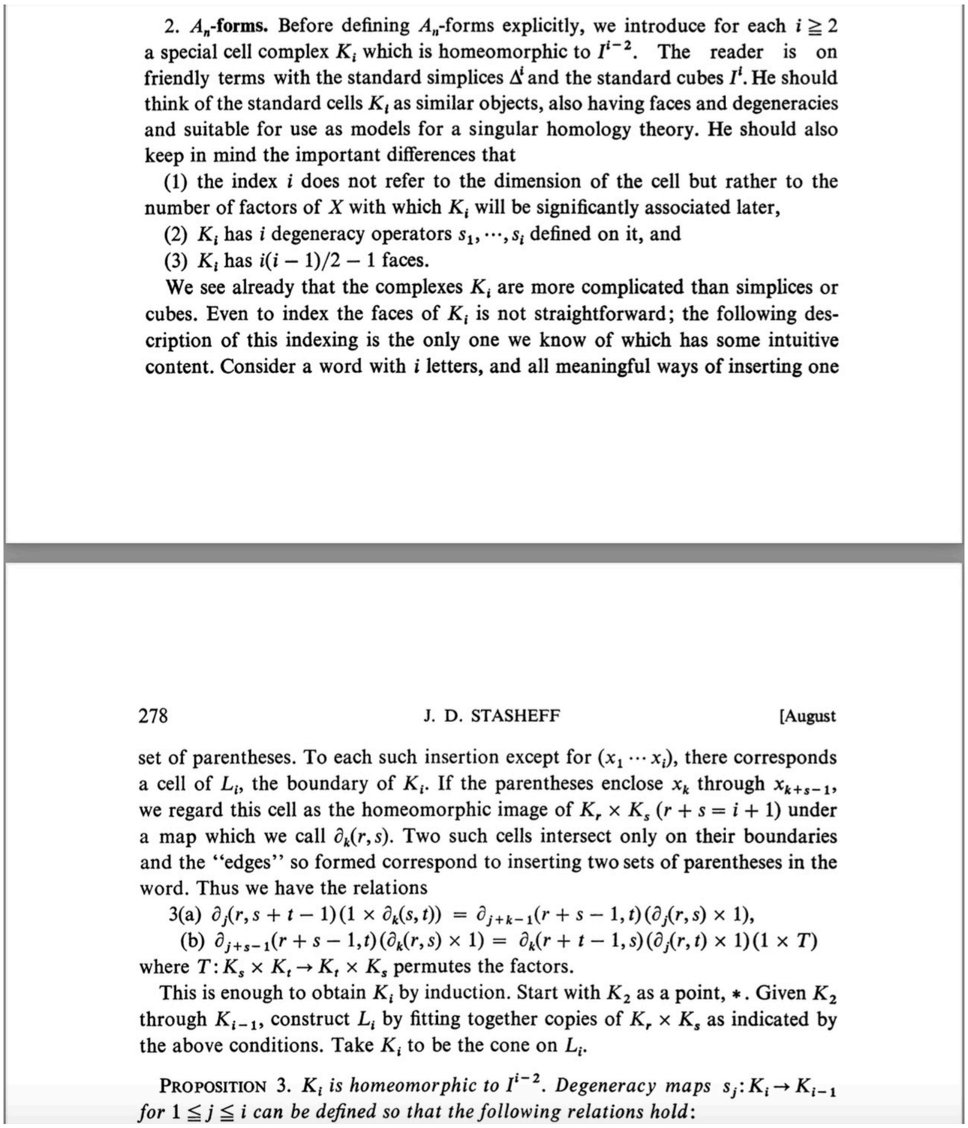
IV.1.1. Stasheff associahedra. You've already seen the Stasheff associahedra in the talks of Catherine and Nate. These are usually presented as K_2 is a point, K_3 is an interval, K_4 is a pentagon, followed by some promises to the reader that generalizations encoding higher versions exist. These promises hold true, but if you were to wonder how exactly to realize these promises yourself, you would be justified – it's not easy.

The big picture to keep in mind is:

The kth Stasheff associahedron K_k is the space of (planar, rooted, stable) metric trees with k leaves. The collection $K_k, k \geq 0$ has an operad structure given by grafting trees.

For those interested, let me show you Stasheff's original writings in Figures IV.1.1.1 and IV.1.1.2, to show how non-trivial it is to set things up. This isn't to intimidate you, but again to illustrate that progress is gradual and slow, and we are lucky to live in times which organize things with the advantage of hindsight. I hope you'll do the same and organize the structures that seem confusing to us today.

[2]The A in A_∞ stands for "associativity." The E in $\mathbb{E}_n$ is supposed to stand for "everything," which was supposed to mean "associative and commutative." So an $\mathbb{E}_n$ algebra was meant to encapsulate something that is both homotopy and commutative, but up to some obstruction presented in dimension n, as we've already seen. Unfortunately, when $n = 1$, there is no commutativity at all, so the etymology of "everything" is a bit misleading. By the way, let me tell you that being associative and commutative certainly isn't everything; but as we've seen, it is at the very least terminal in the theory of operads.



2. **A_n-forms.** Before defining A_n-forms explicitly, we introduce for each $i \geqq 2$ a special cell complex K_i which is homeomorphic to I^{i-2}. The reader is on friendly terms with the standard simplices Δ^i and the standard cubes I^i. He should think of the standard cells K_i as similar objects, also having faces and degeneracies and suitable for use as models for a singular homology theory. He should also keep in mind the important differences that

(1) the index i does not refer to the dimension of the cell but rather to the number of factors of X with which K_i will be significantly associated later,

(2) K_i has i degeneracy operators $s_1, \cdots, s_i$ defined on it, and

(3) K_i has $i(i-1)/2 - 1$ faces.

We see already that the complexes K_i are more complicated than simplices or cubes. Even to index the faces of K_i is not straightforward; the following description of this indexing is the only one we know of which has some intuitive content. Consider a word with i letters, and all meaningful ways of inserting one

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set of parentheses. To each such insertion except for $(x_1 \cdots x_i)$, there corresponds a cell of L_i, the boundary of K_i. If the parentheses enclose x_k through x_{k+s-1}, we regard this cell as the homeomorphic image of $K_r \times K_s$ $(r+s=i+1)$ under a map which we call $\partial_k(r,s)$. Two such cells intersect only on their boundaries and the "edges" so formed correspond to inserting two sets of parentheses in the word. Thus we have the relations

3(a) $\partial_j(r, s+t-1)(1 \times \partial_k(s,t)) = \partial_{j+k-1}(r+s-1,t)(\partial_j(r,s) \times 1)$,

(b) $\partial_{j+s-1}(r+s-1,t)(\partial_k(r,s) \times 1) = \partial_k(r+t-1,s)(\partial_j(r,t) \times 1)(1 \times T)$

where $T: K_s \times K_t \to K_t \times K_s$ permutes the factors.

This is enough to obtain K_i by induction. Start with K_2 as a point, $*$. Given K_2 through K_{i-1}, construct L_i by fitting together copies of $K_r \times K_s$ as indicated by the above conditions. Take K_i to be the cone on L_i.

PROPOSITION 3. *K_i is homeomorphic to I^{i-2}. Degeneracy maps $s_j: K_i \to K_{i-1}$ for $1 \leqq j \leqq i$ can be defined so that the following relations hold*:

FIGURE IV.1.1.1. Excerpt 1. James Dillon Stasheff, "Homotopy Associativity of H-Spaces. I." Transactions of the American Mathematical Society Vol. 108, No. 2 (Aug., 1963), pp. 275-292.

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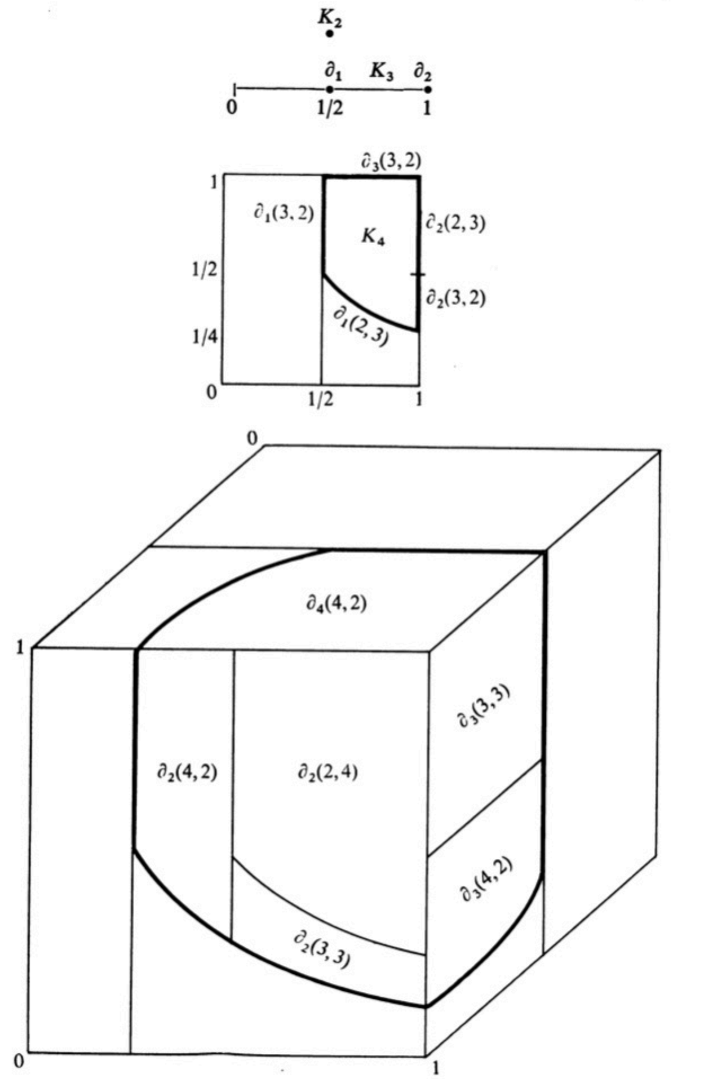

FIGURE 18

FIGURE IV.1.1.2. Excerpt 2. James Dillon Stasheff, "Homotopy Associativity of H-Spaces. I." Transactions of the American Mathematical Society Vol. 108, No. 2 (Aug., 1963), pp. 275-292.

Stasheff's implication about who his readers were and would be does not withstand the test of time. (It also probably failed the test of truth in his own time.)

Returning to our thread, let's recognize that Stasheff himself says that the "only" intuitive reason for the indexing he has is through the notion of parenthesizing variables. This is very satisfying from the perspective of what associativity does capture (relating different parenthesizations); but somewhat clunky to formalize. Let's also look at his drawing of K_4; it is highly non-symmetric, and the "pentagonal" structure only becomes visible after marking an edge. This is evidence that it's hard for members of the community to see good ways to organize things when we first see them.

Remark IV.1.1.3. This is part of a theme in math; often, the most natural way to discover an idea is not the most natural way to organize or describe the thing you discover. If I get to where I want to get to in this lecture, I might be able to describe why this is true for the most common definition of the Fukaya category.

Remark IV.1.1.4. The "presentation" of this 1963 paper occurred in 1959; May's book on geometry of iterated loop spaces didn't come out until 1972. You can see that 10 years of hindsight, with help from people with a talent for vision or for hard work, can improve greatly the way a community views something. (Keep that in mind if you look over these notes in 10 years.)

Anyhow, back to our main narrative. I am going to follow the trend of hand-waving the definition of Stasheff associahedra in the spoken lecture. See the rest of the written notes for my favorite model. The examples we will keep in mind are still

$$K_2 \cong *, \qquad K_3 \cong [0,1], \qquad K_4 \cong \text{pengaton}.$$

As a reminder, K_2 is the "space" of ways a planar tree with two incoming vertices and one root can deform. One thinks of this as just a point. K_3 is the space of ways in which you can deform planar trees with three incoming vertices (and one root). This is an interval.

Obligatory picture here.

You should think of a deformation of a tree as given by changing the length of internal edges from 0 (i.e., the edge collapses) to ∞ (the maximal length an edge is allowed to be), inclusive.

Likewise, K_n is the space of n-incoming-vertices planar trees. K_4 is a pentagon.

Remark IV.1.1.5. For $n \geq 2$, each K_n is homeomorphic to a $(n-2)$-dimensional disk. One can think of the center of this disk as the corolla with n leaves – i.e., a tree with no internal edges.

Obligatory picture here.

IV.1.2. The operadic structure and the product-boundary observation. Let's see some of the operad structure because I want to make a point. Suppose you have an element

$$(S, T_1, \ldots, T_l) \in K_l \times K_{j_1} \times \ldots K_{j_l}.$$

That is, S is a tree with l leaves, and the T_i are trees with j_i leaves. Because there are l such T_i trees, we can graft[3] these trees to the leaves of S to obtain a new tree, which we denote by

$$S \circ (T_1, \ldots, T_l) \in K_{j_1 + \ldots + j_l}.$$

There is one non-obvious element to this grafting. When the ith tree T_i has 2 or more leaves (i.e., when $j_i \geq 2$), the root of T_i becomes a vertex in $S \circ (T_1, \ldots, T_l)$ that has an internal edge emanating from it. We declare the length of this internal edge to be ∞ in defining $\circ$.

WARNING IV.1.2.1. Here, it's important that $K_1 = *$ has a unique element we think of as "degenerate trees with no edges" where the leaf is the root. We also think of $K_0 = *$ as a black hole "tree"; when we graft it to a leaf, the leaf (and the edge emanating from it) vanishes entirely.

Pictures of composed trees

The point I want to make is that the composition of (non-0-ary and non-1-ary) trees always has *at least one internal edge of length* ∞. In other words, the non-trivial operad compositions always have image in the *boundary* of an associahedron. (The interior of the associahedron only parametrizes trees with internal edges having finite length.) And in fact, in a way we can make precise, the boundary of an associahedron is covered exactly by all possible compositions.

In other words, the way that Stasheff defined K_n as a cone on the the union of a bunch of products is compatible with the operad structure itself; so you can flip his definition on its head: K_n is defined inductively by freely writing possible compositions (to obtain ∂K_n), then declaring there is exactly one way to homotope them all (by coning off ∂K_n).

IV.1.3. Cellular chains on the associahedra. Now, let me make just two choices: An orientation for K_2 (i.e., a plus or a minus sign to associated to a point), along with a sign convention equivalent to the *Koszul sign rule* that I will not get into.

It is a lemma that these two choices alone are enough to orient each K_n coherently. In particular, we can define a cellular chain complex over $\mathbb{Z}$. By abuse of notation, I will write $C_*(K_n)$ for these chain complexes.

EXAMPLE IV.1.3.1. Let's spell out the cellular chain complex in some easy examples. I use the base ring $\mathbb{Z}$; if you want a different base ring, change every instance of $\mathbb{Z}$ to your base ring R.

[3]This means we identify the root of T_i with the ith leaf of S. But, depending on the model of tree you take – some people want there to be exactly one edge entering a root – this means identifying the root edge of T_i with the ith leaf edge of S.

(1) We have $C_*(K_0) \cong C_*(K_1) \cong \mathbb{Z}$.
(2) $C_*(K_2) \cong \mathbb{Z}$, a chain complex concentrated in degree 0. (This is the cellular chain complex of a point.)
(3) $C_*(K_3) \cong (\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z})$, a chain complex of rank 1 in degree 1, and rank 2 in degree 0. One can think of the differential as $a \mapsto (a, -a)$ after appropriate orientation of the edge and the boundary vertices of K_3.
(4) $C_*(K_4)$ is congruent to a chain complex of rank 1 in degree 2 (corresponding to the single pentagonal face), of rank 5 in degree 1 (corresponding to the 5 edges of the pentagon), and of rank 5 in degree 0 (corresponding to the 5 vertices of the pentagon).

It follows from Stasheff's inductive definition, and the fact that the gluing of boundary strata is nice, that the operad composition maps are cellular. (This is a non-trivial property.) In particular, the collection

$$C_*(K_n), \qquad n \geq 0$$

forms an operad in chain complexes.

Remark IV.1.3.2. That the collection $C_*(K_n)$ form an operad in chain complexes would be immediate if C_* were singular chains; I emphasize instead that these are cellular chains for a particular cell decomposition of K_n. This is what makes this operad in chain complexes so small.

Remark IV.1.3.3. The Stasheff associahedra are some mysterious looking planar operad in spaces; but by definition, each K_n is contractible. (Stasheff defines K_n as a cone, after all.) In particular, the unique map from this planar operad to the terminal planar operad – otherwise known as $\mathcal{A}$ss (as a planar operad) – is an equivalence. So you should rest assured knowing that whatever operad the Statsheff associahedra encode, they encode algebras that are associative up to homotopy. Or, by taking the free (symmetric) operad on the associahedra, again the map to $\mathcal{A}$ss (as an operad) will be an equivalence of operads. The same is true of $\mathbb{E}_1$, so indeed both the operad of associahedra and the $\mathbb{E}_1$-operad are equivalent (to the associative operad).

Definition IV.1.3.4 (A particular definition of the A_∞-operad in chain complexes). We say that the planar operad $\{C_*(K_k)\}_{k\geq 0}$, with composition induced by the planar operad structure on $\{K_k\}_{k\geq 0}$, is "the" A_∞-operad (in chain complexes). We will denote it by

$$A_\infty(k) := C_*(K_k).$$

Warning IV.1.3.5. This is a planar operad, not an operad. Also, "the" is a bit presumptuous because there are many planar operads equivalent to this model, and many deserve to be called A_∞. But this particular cellular model has proven remarkably useful for formulas, so please let it slide.

We will see in Section IV.3 how this particular model recovers the "usual" formulas for A_∞-algebras in chain complexes that most algebraists and symplectic geometers use.

IV.2. The compactification of the moduli of disks

Compactifications are choices. The Gromov compactness theorem (referenced in Catherine and Nate's talks) tells us that – for analytic reasons – there is a natural compactification of moduli of disks to consider.

Let's be slightly more concrete. Σ be the unit disk, but with $k+1$ boundary points removed. This Σ inherits a complex structure (as a complex manifold with real analytic boundary) from D^2, the usual unit disk. Inherent in our notation will also be a distinguished boundary marked point z_0 among the $k+1$ points we removed; think of this as the "outgoing non-compact boundary" of Σ.

For historical reasons and to relate to the associahedra, we will consider the cases[4] $k \geq 2$ from here on.

There is a $(k+1)$-dimensional space of choices of $k+1$ marked points on a circle. On the other hand, the holomorphic automorphism space of D^2 is $PSL(2,\mathbb{R})$ – three-dimensional. So the moduli of Σ obtained by removing $k+1$ marked points is $(k-2)$-dimensional.

Notation IV.2.0.1. We let

$$\mathcal{R}_{k+1}$$

denote the moduli of holomorphic manifolds (with analytic boundary) obtained by removing $k+1$ boundary marked points from D^2. (This follows the notation of Seidel's book[5], except the $k+1$ is a superscript there.)

Remark IV.2.0.2. It is not hard to see that $\mathcal{R}_{k+1}$ is an *open* manifold – Exercise **??**. Indeed, to see that $\mathcal{R}_{k+1}$ is not compact, you could imagine that two of the marked points collide, or worse yet, that N marked all approach the same point on the unit circle at various rates.

The usual Gromov compactification tells us to do the following: Let us create a stratum in our compactification for what happens when a group of points are actually allowed to collide – or, from another perspective, "run off to infinity" with respect to all other points. In this stratum will be some moduli of *nodal* disks – that is, holomorphic disks $\bigcup_\alpha D_\alpha$, each equipped with $k_\alpha \geq 1$ incoming marked points and one outgoing marked points, together with an identification of the outgoing vertices of some of the D_α with incoming vertices of some $D_{\alpha'}$. (This description is a bit vague;

[4]It is a pleasant surprise (unwritten and unpublished work of Lurie-Tanaka) that when you include the $k=1$ case, the resulting stack classifies planar operads.

[5]Paul Seidel, Fukaya Categories and Picard-Lefschetz Theory (Zurich Lectures in Advanced Mathematics). European Mathematical Society (EMS) 2008.

you might see how to make the description more precise later in the lecture.) We interpret this as a codimension 1 stratum.

Concretely, what if some consecutive collection of points $z_i, \ldots, z_{i+j}$ – with[6] $j \geq 1$ – all become "farther and farther away" from all other points? (You could interpret this as the cross ratio of $z_{i-1}, z_i, z_{i+j}, z_{i+j+1}$ going to infinity.) We introduce a stratum wherein an element is a pair of disks; one disk labeled by points

$$z_0, \ldots, z_{i-1}, w, z_{i+j+1}, \ldots, z_k$$

and the other disk labeled by points

$$w, z_i, \ldots, z_{i+j}$$

where, as indicated, we consider these two disks as glued along w.

We need a picture, though a reader could certainly draw this picture on their own.

We repeat this process to create higher and higher codimension strata, until we construct the codimension $k-2$ strata (hence all 0-dimensional corner strata). Though this description is vague, we set:

Notation IV.2.0.3 (The compactified moduli space of disks with boundary punctures). We let

$$\overline{\mathcal{R}}_{k+1}$$

denote the compactification of $\mathcal{R}_{k+1}$ just described.

Remark IV.2.0.4. I want to be honest with you that I have only described the *set* $\overline{\mathcal{R}}_{k+1}$. The topology is a bit more subtle to pin down. Indeed, many works just tell you what convergent sequences are in this space, and such a definition forces you to be beholden to using sequences in every argument.

Finally, there is an operadic structure on the collection of spaces $\overline{\mathcal{R}}_{k+1}$, given by gluing disks along marked points to produce nodal disks with boundary marked points.

Picture.

IV.2.1. The isomorphism of moduli spaces. The following is the main hint that the symplectic study of holomorphic disks outputs A_∞-structures for chain complexes[7]. If visualizing either of K_k or $\overline{\mathcal{R}}_{k+1}$ was difficult, this theorem allows you to pick your favorite.

Theorem IV.2.1.1. For every $k \geq 2$, there exists a homeomorphism

$$\overline{\mathcal{R}}_{k+1} \to K_k$$

to the kth Stasheff associahedron.

[6]This $j \geq 1$ condition is, as it turns out for enriching spectra over Floer theory, incredibly unnatural.

[7]For spectra, both $\overline{\mathcal{R}}_{k+1}$ and K_k should be replaced by appropriate stacks that see moduli stacks of broken *strips* on the holomorphic side, and of broken trees on the other.

Moreover, one can choose these homeomorphisms to be maps of planar operads.

Remark IV.2.1.2. The only way in which we will use the theorem is to convert the algebra of degenerating disks into m_k operations at the chain level. Strictly speaking, one could have had this entire discussion without the Stasheff associahedra, but that's like teaching you about modes of transportation without teaching you that we have bicycles.

IV.3. The A_∞-relations

IV.3.1. A_∞-categories. Before we go on: We have talked about the notion of algebras over operads. It turns out there is a notion of a category over an operad, as well. Let me hand-wave the definition. Throughout, I will cut through the set-examples and go straight to the context of vector spaces, and more generally chain complexes.

Roughly speaking, what does it mean to give a vector space V the structure of an $\mathcal{O}$-algebra? The most important data are: For every element $x \in \mathcal{O}(k)$, a k-linear map

$$m_x : V \otimes \ldots \otimes V \to V.$$

For the m_x to define an algebra structure, we just need to check some properties.[8]

Remark IV.3.1.1. To generalize this definition from "algebra" to "category," the (often unhelpful) slogan is that a category is just an algebra with one object. So in what follows, make sure to consider the case when the collection of objects $\{X_\alpha\}$ is just a one-element set containing one thing, which we'll call X. Then the endomorphism space of X plays the role of V above.

Now, given a collection of objects $\{X_\alpha\}$ and a collection of vector spaces $V_{\alpha,\beta}$ associated to each pair of objects, one could ask for the data

$$m_x^{\alpha_1,\ldots,\alpha_k} : V_{\alpha_{k-1},\alpha_k} \otimes \ldots \otimes V_{\alpha_0,\alpha_1} \to V_{\alpha_0,\alpha_k}.$$

Interpreting each $V_{\alpha,\alpha'}$ as a "collection of morphisms" from X_α to $X_{\alpha'}$, each m_x could thus be thought of as a formula for a k-fold composition of morphisms.

Example IV.3.1.2. For $k = 2$, an element $x \in \mathcal{O}(2)$, for every triplet of objects X_0, X_1, X_2, defines a map

$$m_x^{\alpha_1,\ldots,\alpha_k} : V_{1,2} \otimes V_{0,1} \to V_{0,2}.$$

Concretely, given elements $f_{i,j} \in V_{i,j}$, one might write this as

$$f_{1,2} \otimes f_{0,1} \mapsto f_{1,2} \circ_x f_{0,1}$$

where $\circ_x$ shoudl be thought of as a "composition" of the two morphisms $f_{0,1}, f_{1,2}$, but a particular composition determined by x.

[8]This is using the definition of operads we have given in the previous lecture.

Definition IV.3.1.3 (Sketch). Fix an operad $\mathcal{O}$ in chain complexes over R (or vector spaces, or sets).[9] An $\mathcal{O}$-*category* is the data of

(a) A collection of objects $\{X_\alpha\}$,
(b) For each pair α, α', a chain complex $V_{\alpha,\alpha'}$ over R (or vector space, or set)
(c) For every $k+1$-tuple of objects $X_0, \ldots, X_k$, a map

$$\mathcal{O}(k) \to \hom(V_{k-1,k} \otimes \ldots \otimes V_{0,1}, V_{0,k})$$

of chain complexes over R (or of vector spaces, or of sets).

These data are required to satisfy some composition and unit relations I do not spell out.

The only example we'll be interested in is when $\mathcal{O} = A_\infty$ is the particular model fo the A_∞ planar operad in chain complexes (Definition IV.1.3.4). You will want to refer to at least the generators of the chain complex as described in Example IV.1.3.1.

So let's spell out what it means to give an A_∞-category $\mathcal{C}$. We first fix a collection of objections, and some chain complexes $V_{X,Y}$ for every pair of objects X, Y. For the Fukaya category – say, in the exact and Calabi-Yau setting – our objects are Lagrangians equipped with tangential structures. The chain complex $V_{L,L'} = CF^*(L, L')$ is the Floer complex. We ignore transversality issues when L and L' are not transverse, to focus on the algebra.

Let us now study the maps

$$A_\infty(k) \to \hom(V_{k-1,k} \otimes \ldots \otimes V_{0,1}, V_{0,k}) \tag{IV.3.1}$$

for every $k+1$ tuple of objects $X_0, \ldots, X_k$.

Example IV.3.1.4 (k=0). For every object X, we have a map

$$A_\infty(0) \cong R \to \hom(R, V_{X,X}) \cong V_{X,X}. \tag{IV.3.2}$$

The last isomorphism follows because we are looking at R-linear maps from R into the R-chain complex $V_{X,X}$. You should think of this map as picking out a unit $R \to V_{X,X}$ for the endomorphisms of X – that is, the image of $1 \in R$ picks out the identity morphism of X. Note that (IV.3.2) is a map of chain complexes; thus it follows that the identity morphism of X is a closed element of the chain complex $V_{X,X}$.

Example IV.3.1.5 (k=1). For every ordered pair of objects X, Y, we have a map

$$A_\infty(1) \cong R \to \hom(V_{X,Y}, V_{X,Y}).$$

The unit axiom for a map of operads will force us to choose this map to pick out exactly the identity map of $V_{X,Y}$; you should think of this as no structure to worry about for the moment.

[9] You may as well take $\mathcal{O}$ to be planar, as the definition will only depend on the planar structure of $\mathcal{O}$; i.e., you can forget the symmetric group actions.

EXAMPLE IV.3.1.6 (k=2). For every ordered triplet of objects X, Y, Z, we have a map

$$A_\infty(2) \cong R \to \hom(V_{Y,Z} \otimes V_{X,Y}, V_{X,Z}).$$

You should think of this as picking out exactly one composition map. For example, if you have $f_{Y,Z} \in V_{Y,Z}$ and $f_{X,Y} \in V_{X,Y}$, the image of $1 \in R$ picks out a map, which is traditionally called m_2:

$$f_{Y,Z} \otimes f_{X,Y} \mapsto m_2(f_{Y,Z}, f_{X,Y}).$$

Note that m_2 depends on X, Y, Z, but we remove it from the notation to reduce clutter.

Now, if m_2 were a composition that is associative on the *nose*, for every quadruple of objects $X_0, \ldots, X_3$ we would desire an equation like

$$m_2(m_2 \otimes \mathrm{id}) = m_2(\mathrm{id} \otimes m_2). \tag{IV.3.3}$$

REMARK IV.3.1.7. Note that (IV.3.3) is an equation of elements in the hom cochain complex

$$\hom(V_{X_2,X_3} \otimes V_{X_1,X_2} \otimes V_{X_0,X_1}, V_{X_0,X_3}). \tag{IV.3.4}$$

Make sure you understand this point.

However, if Mother Nature were to obstruct you and give you an m_2 that is *not* associative, but only associative up to chain homotopy, what would you expect? You would expect that there is some degree 1 element H (H for homotopy) for which

$$\delta H = m_2(m_2 \otimes \mathrm{id}) - m_2(\mathrm{id} \otimes m_2). \tag{IV.3.5}$$

This is an equation inside the chain complex (IV.3.4). Lo and be hold, we will find exactly this by analyzing the $k = 3$ case.

EXAMPLE IV.3.1.8 (k=3). For every ordered quadruple $X_0, X_1, \ldots, X_3$, let's denote $V_{i,j} = V_{X_i,X_j}$ because Hiro is lazy and space is precious. If $\mathcal{C}$ is to be an A_∞-category, we have a map

$$A_\infty(3) \to \hom(V_{2,3} \otimes V_{1,2} \otimes V_{0,1}, V_{0,3}). \tag{IV.3.6}$$

Recall from Example IV.1.3.1 and the Definition of A_∞ (Definition **??**) that $A_\infty(3)$ is a chain complex with two degree 0 generators, and a single degree 1 generator (using homologial grading, not cohomological). The differential is the usual differential realizing two points (corresponding to the degree 0 generators) as the signed boundary of an oriented edge (the degree 1 generator).

By examining the composition maps

$$A_\infty(2) \otimes A_\infty(2) \otimes A_\infty(1) \to A_\infty(3)$$

and

$$A_\infty(2) \otimes A_\infty(1) \otimes A_\infty(2) \to A_\infty(3)$$

(which, at the level of K_3, represent the two different three-leaved trees that are not corolla), the map (IV.3.6) must pick out exactly the elements

$$m_2(m_2 \otimes \mathrm{id}) \qquad \text{and} \qquad -m_2(\mathrm{id} \otimes m_2).$$

(This utilizes a sign convention; you might end up with a different sign – you are fine so long as the two operators appear with opposite signs.)

And indeed, the degree 1 element of $A_\infty(3)$ picks out exactly an element H as in (IV.3.5). We see (IV.3.5) is obtained by noting that (IV.3.6) is a chain map.

We want to pick a less ad-hoc notation than "H".

Notation IV.3.1.9 (m_k). We let m_3 denote the image of the 1-dimensional oriented generator of $A_\infty(3)$. More generally, we let

$$m_k$$

denote the image of the oriented top-dimensional (i.e., degree $k-2$) generator of $A_\infty(k)$ under the map (IV.3.1).

So Example IV.3.1.8 tells us that

$$\delta m_3 = m_2(m_2 \otimes \mathrm{id}) - m_2(\mathrm{id} \otimes m_2)$$

inside the cochain complex

$$\hom(V_{2,3} \otimes V_{1,2} \otimes V_{0,1}, V_{0,3}).$$

Unwinding the definition of the differential δ of this hom complex, we find

$$m_1^{V_{0,3}} m^3 + m_3(m_1 \otimes \mathrm{id} \otimes \mathrm{id}) - m_3(\mathrm{id} \otimes m_1 \otimes \mathrm{id}) + m_3(\mathrm{id} \otimes \mathrm{id} \otimes m_1) = m_2(m_2 \otimes \mathrm{id}) - m_2(\mathrm{id} \otimes m_2).$$

Re-writing every term onto one side, we find the usual A_∞ relation for $k = 3$:

$$0 = \sum_{a+b+c=k} (-)^{\text{some signs}} m_{a+1+c}(\mathrm{id}^{\otimes a} \otimes m^b \otimes \mathrm{id}^{\otimes c}). \tag{IV.3.7}$$

And if you plug in elements of the $V_{i,j}$, writing $f_{i,j} \in V_{i,j}$ we find the formula

$$\begin{aligned} 0 = & m_1 m_3(f_{23}, f_{12}, f_{01}) \\ & \pm m_3(m_1 f_{23}, f_{12}, f_{01}) \pm m_3(f_{23}, m_1 f_{12}, f_{01}) \pm m_3(f_{23}, f_{12}, m_1 f_{01}) \\ & \pm m_2(m_2(f_{23}, f_{12}), f_{01}) \pm m_2(f_{23}, m_2(f_{12}, f_{01})) \end{aligned}$$

where the $\pm$ are signs I am not working out here.

It's up to you whether you like what I just went through. I tried to concretely write out the formulas that everybody writes, but from first principles – to assure you that the formulas have origins and the origins are articulable using concrete ideas.

But now, we can unwind the definition of what it means to be an A_∞-algebra in chain complexes, to arrive at the following (which you have probably seen before):

Definition IV.3.1.10 (A_∞-category). An A_∞ category (in chain complexes) is the data of

(a) A collection $\{L_\alpha\}$ which we will call a collection of *objects*. Each element L_α of this collection will be called an object.
(b) For every ordered pair of objects L_0, L_1, a chain complex V_{L_0,L_1}. As a matter of notation, we will write the differential of this chain complex as m_1.
(c) For every ordered $(k+1)$-tuple of objects $L_0, \ldots, L_k$, and operation

$$m_k : V_{k-1,k} \otimes \ldots \otimes V_{1,2} \otimes V_{0,1} \to V_{0,k}[k-2]$$

(of degree $k-2$, as indicated,

where these data are required to satisfy the so-called A_∞ *relations* (IV.3.7).

There are various unit conditions one could require of this data, which we do not spell out.

IV.4. Why did Hiro spend time spelling out what an A_∞-category is?

The reason is that, modulo details, the Fukaya category of a symplectic manifold is an A_∞-category. There are various geometric caveats:

- If certain obstructions involving $2c_1(TM)$, and the $\det^2$ maps for Lagrangians, do not vanish, then the Fukaya category is not $\mathbb{Z}$-graded, but is $\mathbb{Z}/n\mathbb{Z}$-graded. One way to think of a $\mathbb{Z}/n\mathbb{Z}$-graded complex is as a single complex equipped with an equivalence from itself to its degree n shift,
- If certain obstructions do not vanish, the Fukaya category may be linear over one base ring, but not another. This has to do with being able to orient moduli spaces.
- If the areas of disks matter (e.g., if counts of disks do not converge unless we only count disks of a given area at a time) then the Fukaya category is linear over the Novikov ring. If you can ignore areas of disks (e.g., in the exact case with nice behavior at infinity) then you can also ignore the Novikov variable and work over your base ring R.
- If your Lagrangians bound holomorphic disks, then the Fukaya category is a *curved* A_∞-category meaning it has a m_0 term. The "right" philospphical way to think about such a thing is as a receptacle for deformation problems. It turns out that you can try to solve a certain algebraic deformation problem for every Lagrangian, and when you can, you'll get a well-defined A_∞-cateogry spanned by Lagrangians that admit a "uniform" solution to this deformation problem.

We ignore these caveats. As a general rule, if $\omega = d\theta$ and if the only Lagrangians you consider are those with $[\theta|_L] = 0 \in H^1(L)$, then you can more or less assume that the Fukaya category is an A_∞-category in the sense of Definition IV.3.1.10.

IV.5. Verifying the A_∞-relations in the Fukaya category

So why should the geometry of Lagrangians, and the way that disks can have boundary on tuples of Lagrangians, even have any kind of algebraic structure (e.g., that of an A_∞-category)?

The most efficient but hand-wavy answer is "We've already seen that the moduli of disks define the A_∞ operad."

It is also a misleading answer. When we prove the A_∞-relations, we will be forced to examine *not* just the way that disks degenerate as in the Stasheff associahedra (which involve no strips), but we must also study the appearance of strips (i.e., disks with only two boundary marked points).

Remark IV.5.0.1. Indeed, the way to make this answer less misleading – which would take us astray – is to recognize that when you include strips into the Stasheff associahedra, you will construct a stack that encodes both planar structures, and ways to do Koszul duality with planar structures.

We would really love Theorem IV.2.1.1 to just "be the explanation" that Fukaya categories are A_∞. But of course, the theorem couldn't possibly be that, for the strip reasons already explained. This is the sense in which natural A_∞ structure (which we'll deduce in just a moment) are not organized or explained in a natural way.

IV.5.1. Review of the m_k operations in the Fukaya category.

So before we go on, let's remember what we're doing. Fix some collection of Lagrangians $L_0, \ldots, L_k$ inside a symplectic manifold M, along with intersection points $y_i \in L_i \cap L_{i-1}$ (including the case "$i = k, i+1 = 0$"). We assume the L_i are in general position for simplicity. Then for any pair, we define the hom chain complex (what we have been calling $V_{i,j}$ earlier) to be

$$CF^*(L_i, L_j)$$

i.e., a chain complex generated by the intersection points, and a differential computed by counting holomorphic strips.

Let me explain the holomorphic disks we're going to count. The case $k = 1$ explains the holomorphic strip case, too.

Notation IV.5.1.1. By a "holomorphic disk" with these boundary conditions, I mean the data of a pair

$$(\Sigma, u)$$

where $\Sigma \in \mathcal{R}_{k+1}$ is a choice of a domain for u – namely, a complex disk with $k+1$ boundary punctures[10] – and

$$u : \Sigma \to M$$

is a smooth map for which the ith deleted point of Σ converges to y_i, where the edge after[11] the ith deleted point has image contained in L_i, and (more

[10]When $k = 1$ or 2, there is a unique Σ

[11]This is with respect to the boundary orientation on Σ, induced by the orientation of Σ (which is complex, hence has an orientation).

or less) u satisfies the differential equation

$$du \circ j_\Sigma = J_M \circ du.$$

Here, du is the derivative of u, j_Σ is the complex structure of Σ, and J_M is a chosen almost-complex structure on M.

As we learned from Nate and Catherine's talks, the space of such pairs $\{(\Sigma, u)\}$ – when we choose things nicely – is a smooth manifold, possibly non-compact, of predictable dimension. Moreover, the "predictable" dimension is computed using numerical invariants one can associate[12] to each $|y_i|$.

The remarkable fact is that these invariants satisfy the equation[13]

$$|y_1| + \ldots + |y_k| = |y_0| \pm (2-k)$$

precisely when the dimension of the space of pairs $\{(\Sigma, u)\}$ *is* zero-dimensional[14]. On the other hand, by Gromov compactness and regularity, when the space of such pairs is zero-dimensional, said space is *already* compact – there are only finitely many such pairs.

So what we then define is a map (this is just a function between two chain complexes; it is not a chain map)

$$m_k : CF^*(L_{k-1}, L_k) \otimes \ldots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_k)[\pm(2-k)]$$

which, on generators, is given by the formula

$$y_k \otimes \ldots \otimes y_1 \mapsto \sum_{y_0, |y_0| = |y_1| + \ldots + |y_k| \pm (2-k)} \sharp\{(\Sigma, u)\} y_0 \tag{IV.5.1}$$

where $\sharp$ refers to the number of points in the space of pairs $\{(\Sigma, u)\}$, counted with sign.[15] Note, importantly, that the degree shift of $2-k$ (which was analytically motivated by the desire for 0-dimensional moduli spaces) corresponds exactly to the degree in which an element m_k ought to live by the definition of A_∞-category (which in turn was operadically/algebraically motivated by the Stasheff associahedron model).

IV.5.2. Verifying the relations: Notation for $\mathcal{M}$. There are various geometric and analytic techniques needed to make the following theorem honestly true as stated.

THEOREM IV.5.2.1. The maps m_k defined by the formula (IV.5.1) satisfy the A_∞ relations (IV.3.7).

---

[12]This typically requires some tangential structures to be chosen on each Lagrangian – such as lifts of the $\det^2$ map.

[13]The $\pm$ is there because of differences in homological versus cohomological conventions – the algebraic convention chosen corresponds to a choice of geometric convention where you think of the y_i as located at incoming or outgoing boundary punctures of Σ.

[14]In the case $k = 1$, the space of such pairs always has an $\mathbb{R}$-action because the J-holomorphic curve equation has an $\mathbb{R}$-symmetry, so we interpret the dimension of the space of pairs to be the dimension obtained after modding out by the $\mathbb{R}$ action.

[15]These signs depend on an orientation of the moduli of such pairs, and this orientation depends on further choices on our Lagrangians of Spin structures.

Remark IV.5.2.2. Here are some of the common "techniques" to make the theorem true. Some people like to equip their Lagrangians with isotopy data (for every tuple of Lagrangians) rendering Lagrangians transverse for each m_k operation. This accordingly needs some choices of coherent deformations of the J-holomorphic curve equation. Others also like to consider a Fukaya category by fixing some countable and general-position collection of Lagrangians once and for all, and ignore unitality/endomorphism issues (which one can do for various reasons – for example, we sometimes set up the Fukaya category as a "directed" category, and then localize with respect to maps induced by Hamiltonian isotopies).

We'll only illustrate why you should believe this theorem for the $k = 3$ A_∞ relations, and when the Lagrangians are in general position.

We don't want to keep saying "space of pairs" (Σ, u) so let's introduce some notation.

Notation IV.5.2.3. Fix $k+1$ Lagrangians $(L_0, \ldots, L_k$ along with intersection points $y - i \in L_i \cap L_{i-1}$ for $i = 0, 1, \ldots, k$. We let

$$\mathcal{M}(y_1, \ldots, y_k; y_0) = \{(\Sigma, u)\}$$

denote the space of pairs as before – Notation IV.5.1.1.

Finally, when we do not specify y_0, we let

$$\mathcal{M}(y_1, \ldots, y_k; -) := \coprod_{y_0} \mathcal{M}(y_1, \ldots, y_k; y_0).$$

As of now, if everything is regular, this is a *non-compact* smooth[16] manifold with dimensions prescribed by a formula involving the $|y_i$. Finally, we let

$$\mathcal{M}(y_1, \ldots, y_k; -)^{\dim = d}$$

denote the component of this manifold that is d-dimensional.

What I'd like to emphasize is that here, we will choose $y_0, y_1, \ldots, y_k$ so that the dimension of $\mathcal{M}(y_1, \ldots, y_k; y_0)$ may not be zero.

Remark IV.5.2.4. To prove the A_∞ relations, one takes advantage of the fact that each $V_{i,j} = CF^*(L_i, L_j)$ has a chosen set of generators. So to verify the relations, one need only check that, each time we plug in inputs $y_1, \ldots, y_k$, for every generator of the codomain chain complex y_0, we see that the y_0 coefficient is zero.

IV.5.3. Verifying the relations, II: The way $\mathcal{M}$ lives over $\mathcal{R}$. We will prove the A_∞-relations by studying the non-compact manifold of dimension 1

$$\mathcal{M}(y_1, \ldots, y_k; -)^{\dim = 1}.$$

[16]We know that each $\mathcal{M}$ is smooth by the inverse function theorem in the setting of Banach manifolds – our regularity assumption tells us that $\mathcal{M}$ is cut out as the intersection of two smooth objects that one should think of as transverse.

Note that I have fixed the y_i for $i = 1, \dots, k$, so by the dimension formula, the $-$ output must necessarily be some point z_0 whose degree $|z_0|$ is 1 off from the outputs y_0 in the A_∞-operations.

By design, there is a forgetful map

$$\mathcal{M}(y_1, \dots, y_k; -) \to \mathcal{R}_{k+1}, \qquad (\Sigma, u) \mapsto \Sigma.$$

What Gromov compactness tells us is that this forgetful map extends to the compactifications,

$$\overline{p} : \overline{\mathcal{M}}(y_1, \dots, y_k; -) \to \overline{\mathcal{R}_{k+1}},$$

and in a predictable way.

Remark IV.5.3.1. We would like to replace the word "predictable" with more precise words, such as "codimension-preserving" and "factorizable." The fact that we cannot say this about p is one way in which the usual $\overline{\mathcal{R}_{k+1}}$ is, as it turns out, an unnatural object for organizing all the data of the Fukaya category.

What Gromov compactness, in the setting we are in, tells us is that one can form a compact space containing $\mathcal{M}(y_1, \dots, y_k; -)$ by adjoining points that look like pairs:

$$(\wedge_j \Sigma_j, u) \tag{IV.5.2}$$

where the first variable is not a single disk, but a wedge of disks, each with at least one incoming and one outgoing marked points. The rule is that the wedge points take place along one incoming marked point of a disk, and one outgoing marked point of another disk. u is a map out of this wedge into your symplectic manifold M, where $u|_{\Sigma_j}$ is the usual kind of map we study in this game.

Most importantly, the wedges of these disks look like mutated Mickey Mouse pictures, but are topologically always contractible (no cycles). They can in fact be drawn as a planar configuration of disks – see Figure IV.5.3.2. In particular, there are "initial" input marked points; i.e., those that are not glued to any outgoing marked point of any other disk. These detect the boundary conditions given by $y_1, \dots, y_k$. There is also a "terminal" output marked point – the lone output that is not glued to any incoming vertex of any disk. The limit of this point is what is dictated by the boundary condition y_0.

Figure IV.5.3.2. Picture of some sample configurations defining $\wedge\Sigma_j$.

And next most importantly, some Σ_j may be *unstable*, in that Σ_j could be a strip – a disk with only two marked points – which has a continuous $\mathbb{R}$ symmetry and is thus not usually considered a stable curve. This is a clear difference from the way we compactified $\mathcal{R}$ into $\overline{\mathcal{R}}$ – there, we only looked at Micky Mouse pictures without strips.

NOTATION IV.5.3.3. We let

$$\overline{\mathcal{M}}(y_1, \ldots, y_k; z_0)$$

denote the space of pairs (IV.5.2). This is the Gromov compactification of $\mathcal{M}(y_1, \ldots, y_k; z_0)$.

Let's draw a picture.

FIGURE IV.5.3.4. Picture of $\mathcal{M}$ before compactifying

Figure IV.5.3.5 is a picture of $\mathcal{M}(y_1, \ldots, y_k; -)^{\dim=1}$. It is a 1-manifold with many connected components, and I've draw it suggestively as living over the pentagon (an associahedron), as though $k = 3$. (Indeed, as we've discussed, there is always a forgetful map from $\mathcal{M}(y_1, \ldots, y_k; -)$ to $\mathcal{R}_{k+1}$.) Some connected components could be circles, but all other connected components are open intervals (by virtue of $\mathcal{M}$ being a manifold). Note that I've drawn these open intervals so that their endpoints might live over the codimension 1 boundary of the pentagon, or live over the interior of the pentagon.

FIGURE IV.5.3.5. Picture of $\mathcal{M}$ after compactifying

What Figure IV.5.3.5 depicts is the compactification, obtained by filling in the "open boundary points" of the intervals. The power of Gromov compactness is that these new points (that fill in the holes) have known forms:

(a) In codimension 1, they are obtained by either forming Mickey Mouse configurations forming one new strip, or they are obtained precisely by proceeding to the codimension 1 boundary of $\overline{\mathcal{R}}_{k+1}$.
(b) Moreover, *all* possible ways of ways of concatenating strips and disks appear in the boundary of $\overline{\mathcal{M}}$.

Informally, the state of affairs may be written as[17]

$$\partial(\overline{\mathcal{M}}^{\dim=1}) \cong \mathcal{M}^{\dim 0} \circ \mathcal{M}^{\dim 0}$$

where $\circ$ indicates the natural way in which we can concatenate pairs Σ, u. A less imprecise way of writing the above is

$$\partial(\overline{\mathcal{M}}(y_1, \ldots, y_k; z_0)^{\dim=1}) = \bigcup_{a+b\leq k; a,b\geq 0} \bigcup_{w} \mathcal{M}(y_{a+1}, \ldots, y_{a+b}; w) \times \mathcal{M}(y_1, \ldots, y_a, w, y_{a+b+1}, \ldots; z_0)$$

[17]I write it in this form to explain why something looking like this came up in Kate Poireier's talk. I also hope that this geometric statement looks a lot like a Maurer-Cartan equation. Indeed, I think the "correct" way to phrase the Fukaya category (over chain complexes or over spectra) is as a category obtained by solving the Maurer-Cartan equation given geometrically from the moduli of disks.

where w runs over all intersection points $w \in L_{a+b-1} \cap L_{a+b}$ rendering the moduli $\mathcal{M}$ on the right to have dimension zero.

If enough structures are chosen so that one can orient $\overline{M}^{\dim=1}$, then we have the usual fact of differential topology that (counted with sign) the boundary of a compact 1-manifold is null. Thus, if one can find concrete chain models (e.g., an appropriate cellular chain complex for the $\overline{\mathcal{M}}$) for which the fudamental classes of $\mathcal{M}(y_1, \ldots, y_k; -)^{\dim=0}$ represent the operations m_k, one recovers the A_∞ equations (IV.3.7) for the inputs $y_1, \ldots, y_k$.

IV.6. Why a compactified moduli space of disks recovers the associahedra (not covered in lecture)

I did not explain how one proves Theorem IV.2.1.1. One sketch is given in a paper of Fukaya-Oh[18], but the ideas there are a bit ad hoc.

I will tell you one way to construct this homeomorphism, at least on the interior of the moduli spaces. The construction is natural enough that I think it will help you convince yourself of what should happen near the boundary.

The map I'm about to show you is a map I learned from Curt McMullen. Some students that I work with are writing a detailed proof of the above theorem using this particular map. All you need to know about is the hyperbolic geometry of the Poincaré disk.

Fix a configuration $z_0, z_1, \ldots, z_k \in S^1$, which we may assume to be ordered with respect to the orientation of the circle (as a boundary to D^2, which is oriented by virtue of being complex). For every consecutive pair of points z_i, z_{i+1} modulo $k+1$, there exists a unique hyperbolic geodesic between the pair. Drawing these geodesics, one obtains a star-shaped region in D^2.

We have a distinguished z_0, so let's use it. Draw all circles tangent to S^1 at z_0, and contained in D^2. Such circles are called *horocycles*. It is a fact of hyperbolic geometry that any two horocycles have a well-defined distance between them. One way to witness this distance is as follows. There is a natural $\mathbb{R}$-action on the space of horocycles (the horocycles are the orbit of a parabolic inside $PSL(2, \mathbb{R})$) and the $\mathbb{R}_{>0}$ part of this action moves all horocycles toward the root z_0; the real number that takes a given horocycle to another is the distance between the two horocycles.

These horocycles foliate the star-shaped region. Consider the leaf space. Concretely, we quotient the star-shaped region by the relation

$$x \text{ and } x' \text{ lie on a connected arc of a horocycle in the star-shaped region.}$$

So $[x]$ – an element of the leaf space – represents an arc of a horocycle. (One horocycle may, after being instersected with the star-shapred region, give

[18]Kenji Fukaya and Yong-Geun Oh, "Zero-loop open strings in the cotangent bundle and Morse homotopy." Asian Journal of Mathematics, Volume 1 (1997), Number 1, Pages: 96 – 180. DOI: https://dx.doi.org/10.4310/AJM.1997.v1.n1.a5

rise to many disconnected arcs.) And the $\mathbb{R}_{>0}$ action preserves the action of moving each $[x]$ in a distinguished direction toward z_0. This exhibits the leaf space as a tree with root z_0 at infinity, and the hyperbolic metric exhibits the tree as a metric tree (there is a real number associated to each internal edge, called "what element of $\mathbb{R}_{>0}$ is required to travel along the whole edge?").

Picture

There are a lot of cool things one could say about this map; there is actually another version that involves doubling Σ to a puncture $\mathbb{P}^1$ that may be better-behaved, but let me not go into it.

IV.7. Broken objects (not covered in lecture)

I made some minor quibbles about how the map $\overline{p} : \overline{\mathcal{M}} \to \overline{\mathcal{R}}$ cannot be made codimension-preserving on the nose (Remark IV.5.3.1).

One way to make $\overline{p}$ codimension preserving is to realize that $\mathcal{R}$ does not just want to compactify; but it wants to be the open part of some stack where disks are allowed to degenerate allowing for strips to appear. Because strips have automorphisms, such a moduli space must naturally be a stack. In fact, the unnatural pattern of $K_1 = *, K_2 = *$ in the associahedra (which breaks the dimension formula of $\dim K_d = d - 2$) is made more natural if one can create an operad where K_1 is the moduli of strips – a copy of "point modulo $\mathbb{R}$," otherwise known as $B\mathbb{R}$.

The most natural way I know of for thinking about Fukaya-categorical structures is thus exploiting a version of the map $\overline{p}$ where the codomain is no longer $\overline{\mathcal{R}}$, but a stacky expansion thereof – a stack of broken trees, or of broken holomorphic disks. This stack, or more precisely, the disjoint union of all these stacks for $k \geq 1$, turns out to be a phenomenal object. It not only receives a natural codimension-preserving and product-structure-respecting map from $\overline{\mathcal{M}}$, it also turns out to encode Koszul duality for planar operads. None of this is written down at present.

But a study of the $k = 1$ case, where one expects to recover associative algebra structures (because the space of 1-ary operations forms an associative algebra) is exposited a bit in the work of Lurie-Tanaka[19].

[19] Jacob Lurie and Hiro Lee Tanaka. "Associative algebras and broken lines." arXiv:1805.09587.

Exercises about A_∞-categories

IV.8. A_∞-categories and their cohomology categories

Definition IV.8.0.1. Suppose that your base ring R is a field, and fix an A_∞-category $\mathcal{C}$ over R. We let

$$H^*\mathcal{C}$$

denote the category with the same objects as $\mathcal{C}$, but where

$$\hom_{H^*\mathcal{C}}(X, Y) := H^* V_{X,Y}$$

where $V_{X,Y}$ is our clunky notation for the morphisms in our A_∞-category.[20]

(a) Show that $H^*\mathcal{C}$ is a (possibly non-unital) category.
(b) What conditions would you impose on $\mathcal{C}$ to make $H^*\mathcal{C}$ unital? (It turns out that there are a few different options; in a higher-categorical sense, whether a category is unital or not is a *property* of the category, so all these definitions turn out to be equivalent under the same framework. Indeed, the simplest definition of unitality is that $H^*\mathcal{C}$ ends up a unital category.
(c) Was it necessary that R is a field?

IV.9. A_∞ versus dg

Verify as much of the following as you like. You can find particular models for everything below in Seidel's book[21], the first chapter.

(a) Let $\mathcal{C}$ be an A_∞ category with A_∞ operations m_k. Suppose all the $m_k = 0$ for $k \geq 3$. Show $\mathcal{C}$ is a dg-category.
(b) Look up the definition of the *opposite* of an A_∞-category.
(c) Look up the definition of the A_∞-category of A_∞-functors between two A_∞-categories.
(d) Any A_∞-category $\mathcal{C}$ has a Yoneda embedding; i.e., a fully faithful functor

$$\mathcal{C} \to \mathsf{Fun}_{A_\infty}(\mathcal{C}^{\mathrm{op}}, \mathsf{Chain}_R).$$

(Warning: This fully-faithfulness is detected at the level of H^*.)

[20] The reason that I used this notation in the notes is to avoid the multiply layers of hom that would appear otherwise.

[21] Paul Seidel, Fukaya Categories and Picard-Lefschetz Theory (Zurich Lectures in Advanced Mathematics). European Mathematical Society (EMS) 2008.

(e) For any dg-category $\mathcal{D}$ and any A_∞-category $\mathcal{C}$, the A_∞-category
$$\mathsf{Fun}_{A_\infty}(\mathcal{C}, \mathcal{D})$$
of A_∞-functors is, in fact, a dg category.

(f) An *equivalence* of A_∞-categories is a functor that induces isomorphisms on all cohomology groups of hom-complexes. Any A_∞-category is A_∞-equivalent to a dg category.

Fukaya category exercise

IV.10. Turning higher-dimensional pictures into 2-dimensional pictures

This is a common trick in the field for visualizing what higher morphisms might be doing in the Fukaya category. It reminds me of one time Eric Zaslow told me "you can just draw everything on the plane anyway."

Let M be a symplectic manifold and fix Lagrangians $L_0, \ldots, L_k$. (I would encourage you to think about the case $k = 1$ first.)

(a) If $k = 1$, let $\gamma_0 = \mathbb{R} \subset \mathbb{R}^2$ and let γ_1 be some curve obtained by considering the line $\ell = \{y = 1\}$, then deforming ℓ in a compact region to droop "below" γ_0 in a transverse way; the resulting γ_1 intersects γ_0 in exactly two points, and is parallel to γ_0 outside a compact set.

Pretending everything goes right, and ignoring the areas of disks, compute the Floer complex

$$CF^*(L_0 \times \gamma_0, L_1 \times \gamma_1)$$

including the differential.

(b) Show that your answer is the mapping cone of the identity morphism from $CF^*(L_0, L_1)$ to itself. (In other words, the complex above is acyclic – this is consistent with being able to Hamiltonian isotope γ_1 to be disjoint from γ_0.)

Remark IV.10.0.1 (Relation to Lagrangian cobordisms). By replacing $L_1 \times \gamma_1 \subset M \times \mathbb{R}^2$ with some Lagrangian *cobordism* from L_1 to L_2, one obtains interesting morphisms from $CF^*(L_0, L_1)$ to $CF^*(L_0, L_2)$. (The Floer complex pairing $L_0 \times \gamma_0$ against the cobordism is again a mapping cone, but of a potentially non-identity morphism.) By treating L_0 as a variable, by the Yoneda lemma, we see that Lagrangian cobordisms give rise to morphisms in the Fukaya category.

This remark, of course, ignores the analytic and geometric questions of when it makes sense to set up a Floer theory for Lagrangian cobordisms.

(c) Further fix curves $\gamma_0, \ldots, \gamma_k \subset \mathbb{R}$; when $k \geq 2$, you can take these to be lines in general position. Understand the m_k operations for the Lagrangians

$$L_0 \times \gamma_0, \ldots, L_k \times \gamma_k \subset M \times \mathbb{R}^2$$

in terms of the m_k operations for the Lagrangians

$$L_0, \ldots, L_k \subset M.$$

More operad exercises

Although the motivation for these courses is to set you up for Fukaya categorical structures enriched over spectra (not chain complexes), the "easier" case is when everything is enriched over chain complexes. A lot of these exercises give you some practice so that the chain complex techniques do start feeling "easier."

IV.11. Koszul sign rule practice

There is a general principle that helps us figure out what signs to place in our formulas: (i) Every symbol has a degree. (ii) Each time you move one symbol past another, you introduce a sign $(-1)^{pq}$ where p and q are the degrees of the symbols you move.

Example IV.11.0.1. We usually say that a multiplication is commutative if xy - yx = 0. You should think of this equation as expressing commutativity when x, y have degree 0.

We will say that a multiplication is commutative in the graded sense if

$$xy - (-1)^{|x||y|}yx = 0$$

where $|x|$ is the degree/grading of x.

Usually, a derivation on a ring is an operator d for which

$$d(x \cdot y) = dx \cdot y + x \cdot dy.$$

More generally, a derivation of degree k is an operator d, taking degree l elements to elements of degree $k + l$, for which

$$d(x \cdot y) = dx \cdot y + (-1)^{k \cdot |x|} x \cdot dy.$$

IV.11.1. Sphere swaps. Let $\sigma : \mathbb{R}^{n+m} \cong \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$ be the map taking

$$(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (y_1, \ldots, y_m, x_1, \ldots, x_n).$$

We have an induced continuous map on one-point compactifications:

$$S^{n+m} \cong \overline{\mathbb{R}^n \times \mathbb{R}^m} \to \overline{\mathbb{R}^m \times \mathbb{R}^n} \cong S^{m+n} = S^{n+m}.$$

Show that this induced map is, homologically speaking, a map of degree $(-1)^{nm}$. That is, fixing an orientation of S^{n+m}, this map is orientation reversing if both n and m are odd, and orientation preserving otherwise.

Remark IV.11.1.1. Topologically, this is the reason that the Koszul sign rule exists; swapping an n-dimensional sphere past an m-dimensional sphere in the natural way reverses orientation in a way dictated by nm.

IV.11.2. cdgas. Fix a vector space V (over some base field k) and a binary operation (i.e., multipliation) $V \otimes_k V \to V$. Recall that this binary operation is called a commutative if for any $a, b \in V$, $a \otimes b$ and $b \otimes a$ are sent to the same element under this operation.

(a) Now suppose V has a grading (e.g., is a graded vector space). Write out the equation expressing commutativity of a binary operation.
(b) Write out the equation expressing associativity of a binary operation. (Hint: You never swap anything.)
(c) We further say that V is a *commutative differential graded algebra*, or cdga, if V is equipped with a degree $+1$ differential d which (i) satisfies $d \circ d = 0$, and (ii) acts as a (degree 1) derivation on the chosen commutative product. Show that for V to be a cdga is the same thing as writing a commutative and associative product $V \otimes_k V \to V$, where now the tensor product is a tensor product of chain complexes.

    (Recall that $U \otimes_k V$ has same underlying vector space as the usual tensor product, but has a differential given by $d(u \otimes v) = du \otimes v + (-1)^u u \otimes dv$.)
(d) Let X be a smooth manifold, and let $\Omega(X)$ denote its smooth deRham forms over $\mathbb{R}$. Show that $\Omega(X)$ – with its usual product, and its deRham differential – is a cdga.

IV.11.3. dg Lie algebras. Fix a binary operation $V \otimes_k V \to V$. Recall that this binary operation is called a Lie bracket if it is antisymmetric, and if it is a derivation of itself. Writing this operation as a bracket, this last statement becomes

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

(a) Now suppose V has a grading (e.g., is a graded vector space). Write out the two equations expressing antisymmetry of a bracket, and that the bracket is a derivation of itself.
(b) We further say that V is a *dg Lie algebra*, or dgla, if V is equipped with a degree $+1$ differential d which squares to zero and acts as a (degree 1) derivation on a chosen Lie bracket.
(c) Suppose that A is a cdga and V is a dg Lie algebra. Exhibit a natural dgla structure on $A \otimes_k V$ (this is the tensor product of chain complexes).
(d) Let X be a smooth manifold, let $\mathfrak{g}$ be the Lie algebra of a smooth Lie group G. Show that the space of $\mathfrak{g}$-valued differential forms on X is a dgla.

Remark IV.11.3.1. The classic example of a Lie algebra is the collection of smooth vector fields on a smooth manifold; these are ways to "deform" the identity diffeomorphism by flowing. More generally, dglas show up all over the place in modern algebra as a way to deform algebraic structures. If you do a later exercise, you'll see that in fact there is a notion of L_∞ algebra, which are like chain complexes with a Lie bracket satisfying the Jacobi identity but only up to higher and higher specific homotopies. These play an important role in encoding, and classifying, deformation problems.

IV.12. Free algebras and coalgebras

For simplicity, let's fix a base field k.

Let $\mathcal{O}$ be any (symmetric) operad in vector spaces over k, and let V be any vector space (or chain complex) over some base field k. It turns out that one can naively define the *free* $\mathcal{O}$-algebra on V to be the chain complex

$$\mathrm{Free}_{\mathcal{O}}(V) := \bigoplus_{n \geq 0} (\mathcal{O}(n) \otimes_k V^{\otimes_k n})/\Sigma_n$$

(a) Exhibit a natural $\mathcal{O}$-algebra structure on $\mathrm{Free}_{\mathcal{O}}(V)$.
(b) Can you write-down a free-forget adjunction between the category of $\mathcal{O}$-algebras, and the category of chain complexes?
(c) When V is a vector space of dimension d (i.e., a chain complex concentrated in degree 0 with Euler characteristic d), compute $\mathrm{Free}_{\mathcal{O}}(V)$ when $\mathcal{O}$ is the associative operad, and when $\mathcal{O}$ is the commutative operad. Is this what you would expect? It may help to first understand the case $d = 0, 1, 2$.

IV.13. Dunn additivity

The Dunn additivity theorem states that the category of E_1-algebras in the category of E_n-algebras is equivalent to the category of E_{n+1}-algebras. Informally, the theorem states that an E_n-algebra is the same thing as a single object with $n+1$ mutually compatible multiplications.

Here, we will see the importance of *spaces* of morphisms, by observing that for sets, E_2 is the same thing as commutativity.

(a) Let G be a connected Lie group. Show that $\pi_1(G)$ is abelian.
(b) Let M be a monoid. (This means M is a set, equipped with an associative operation $m : M \times M \to M$ admitting a unit $u : * \to M$.) Note that this makes $M \times M$ into a monoid as well, by declaring

$$(M \times M) \times (M \times M) \to M \times M, \qquad ((a, b), (c, d)) \mapsto (ac, bd).$$

Assume that you can give M another multiplication $\mu : M \times M \to M$ for which μ is a map of monoids, and for which μ is associative and unital. (In other words, suppose you can exhibit M as a monoid in the category of monoids.) Show that μ and m must be equal, and that both are in fact commutative operations.
(c) Can you use part (b) to prove part (a)?

More practice with $\mathbb{E}_n$ operads

IV.14. Basic computations in $C_*\mathbb{E}_n$

Fix a base ring R.

(a) Compute that the homology of the 1-ary space $\mathbb{E}_n(1)$, and prove that the homology is just R in degree 0. (Hint: Show that $\mathbb{E}_n(1)$ is contractible.)
(b) Compute that the homology of the 2-ary space $\mathbb{E}_n(2)$ is the homology of the sphere of dimension $n-1$.
(c) Conclude that, up to homotopy, if V is given the structure of an $\mathbb{E}_n$-algebra, then the homology of V is endowed with two binary operations – one of degree 0, and another of degree $n-1$.
(d) (*) Assume R has characteristic 0 and assume $n \geq 2$. Show that the degree $n-1$ operation on the homology of an $\mathbb{E}_n$-algebra V defines a (graded) Lie bracket of degree $n-1$.
(e) (*) Assume R has characteristic 0. Show that the degree $n-1$ operation acts as a (graded) derivation on the degree 0 operation.

IV.15. Framed $\mathbb{E}_n$

The *framed* $\mathbb{E}_n$-operad has k-ary space of operations homotopy equivalent to the space of all smooth embeddings of $[0,1]^{\amalg k}$ into $[0,1]$.

(a) Show that $\mathbb{E}_n(1)$ is homotopy equivalent to $O(n)$.
(b) Further, by computing composition of 1-ary operations, show that any framed $\mathbb{E}_n$-algebra is equipped with a (homotopical) action of the orthogonal group.

IV.16. BV algebras and Gerstenhaber algebras

Let's work over a field R of characteristic 0. We let $H_*(\mathbb{E}_2^{fr})$ be the homology of the framed $\mathbb{E}_2$ operad.

(a) Suppose A is a chain complex that is an algebra over chains of the $\mathbb{E}_2$ operad. Show that the homology of A has the structure of a Gerstenhaber algebra.
(b) Suppose A is a chain complex that is an algebra over chains of the framed $\mathbb{E}_2$ operad. Show that the homology of A has the structure of a BV algebra.

Remark IV.16.0.1. What the above exercises suggest is – whenever you have a Gerstenhaber algebra, you can ask whether it secretly arises as an

$\mathbb{E}_2$ algebra. Remember that being an $\mathbb{E}_2$ algebra is better structure; there is more information to be gained than just homology.

The same holds for BV algebras – when you have a BV algebra, you can ask whether it arises as an algebra over the framed $\mathbb{E}_2$-operad.

But the definition of $\mathbb{E}_2$, and framed $\mathbb{E}_2$, are rather geometric. Thus, for something like string topology (which is geometric in origin), one is excited to go ahead and try realizing some string topology operations as arising from framed $\mathbb{E}_2$ structures.

REMARK IV.16.0.2. Algebraic invariants like Hochschild cochains are known to have $\mathbb{E}_2$-structures, while simultaneously satisfying some algebraically articulable universal properties (like classifying deformations). This is strong evidence of the *utility* of being comfortable both with the topological/homotopy-theoretic intuitions of $\mathbb{E}_2$-algebraic structures, and the algebraic motivations for studying things that end up having such structures.

The most satisfying explanation for why Hochschild cochains have an $\mathbb{E}_2$-algebra structure is Dunn additivity: Hochschild cochains compute endomorphisms of the identity functor. But such a collection has two multiplications – one by being endomorphisms of something, and the second because the identity can compose with itself. (People working with 2-categories would say that Hochschild cochains ought to have both a horizontal and a vertical composition.)

LECTURE V

Ways operads (should) show up in Lagrangian Floer theory, II: Moduli of Riemann surfaces and framed $\mathbb{E}_2$ algebras

Last time we tried to see a couple connections:

(i) A connection between trees and disks: The associahedra $\{K_n\}$ are isomorphic to a moduli of boundary-nodal holomorphic disks $\{\overline{\mathcal{R}}_{k+1}\}$.
(ii) A connection between compactifications of moduli of J-holomorphic disk maps and the algebra of A_∞-algebras: The Fukaya category is an A_∞ category.

My complaints about $\overline{\mathcal{R}}$ being blind to strips notwithstanding, there is something very satisfying about all this – the geometry informed the algebra. Most satisfying was that in step (ii), we *used* the geometry of how J-holomorphic objects can map to a symplectic manifold to take advantage of our intuition from (i).

Today, we're going to see that we are still very much in Step (i) for most higher-algebraic structures we want to witness in Fukaya categories, in string topology, and so forth. Indeed, we're going to talk about how certain string topology operations can now be recovered algebraically as purely formal statements about how certain moduli spaces act on certain categorical invariants.[1] But we do *not* know yet, as a community (as far as I understand) how to exhibit these algebraic structures for Fukaya categories in a manner similar to (ii) above – i.e., efficiently and directly from how J-holomorphic disks behave in symplectic manifolds.

Really, this talk is way too ambitious. I want to make clear why we think string topology has anything to do with Fukaya categories, and then illustrate that the moduli of Riemann surfaces has an absurdly important role to play in *algebra*. The moduli of Riemann surfaces was already interesting enough from the perspective of complex/algebraic/smooth topology. That it plays such an important role in higher algebra is a genuine miracle.

[1] We'll try to see various guises of this in factorization homology, in a famous theorem of Costello, and a finally-released pre-print of Kontsevich-Takeda-Vlassopoulos.

V.1. An outline for the talk

"The moduli of holomorphic disks" is the beginnings of some "Deligne-Mumford type[2] moduli space" of all Riemann surfaces (possibly with boundary).

On the other hand, we saw from Kate's talks that some interesting structures in string topology have actions by the moduli of Riemann surfaces. Since the very definition of the Fukaya category involves studying moduli of *maps* out of certain Riemann surfaces (disk), is there some connection between these two?

The answer is yes, when the symplectic manifold is a cotangent bundle with some extra tangential structures that we will not go into. In this talk I will try to explain how:

- The homology of the free loop space – one of the basic groups having interesting operations in string topology – arises as a basic algebraic invariant of Fukaya categories. Namely, by a combination of Abouzaid's theorem and Goodwillie's theorem, we witness $H_*\mathcal{L}Q$ as the Hochschild homology of the wrapped Fukaya category of T^*Q. The S^1 action on both are compatible under this isomorphism. At this point, we are now content asking if, or which of, the string topology operations are formal consequences of purely categorical structures.
- Hochschild cohomology of any linear category has a Gerstenhaber algebra structure – this is because the Hochschild cochain complex is an $\mathbb{E}_2$-algebra. In the setting of the wrapped Fukaya category of T^*Q for Q compact, it turns out that we can witness Hochschild homology as the dual of Hochschild cochains:

$$HH_* \cong (HH^*)^\vee[\pm d] \tag{V.1.1}$$

  up to a shift of degree $d = \dim Q$. As a result, we see that HH_* inherits the structure of a dual of a Gerstenhaber algebra. This already gives us string topology operations on $H_*\mathcal{L}Q$ of a "Gerstenhaber coalgebra."
- Seven months ago, a pre-print[3] of Kontsevich-Takeda-Vlassopoulos (which had circulated in some form for years) was released on the arXiv, and a new version was put up seven days ago (July 12th, 2022). They explain how any category with a "pre-Calabi-Yau" structure has a Hochschild chain complex admitting the action of a PROP computing cohomology of the moduli of Riemann surfaces.

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[2] In what follows, we will also consider strip in our moduli spaces, so these will not – strictly speaking – be Deligne-Mumford in the sense of having only finite automorphism groups.

[3] Maxim Kontsevich, Alex Takeda, Yiannis Vlassopoulos. "Pre-Calabi-Yau algebras and topological quantum field theories." arXiv:2112.14667

> As hinted at by the conjectures Kate told us about, the key ingredient is actually a combinatorial model for describing the moduli of Riemann surfaces in an algebraically usable fashion.

Remark V.1.0.1. It is known that the wrapped Fukaya category of T^*Q is (up to choice of certain tangential structures) a homotopy invariant of Q. The isomorphism V.1.1 depends on a choice of orientation (really, a Poincaré pairing). Thus, all of the string topology operations that are articulable using formal, categorical constructions must also be a "weak" invariant, in that they only depend on data like homotopy type and orientation of M.

On the flip side, the string topology operations known to be "strong" invariants (i.e., that can distinguish the homeomorphism or diffeomorphism classes of manifolds of the same homotopy type) must *not* be invariants that can be articulated formally from categorical constructions beginning with the wrapped Fukaya category.

Remark V.1.0.2. Let me warn you that the results about the wrapped Fukaya category of T^*M, stated above, have to do with a version of the wrapped Fukaya category that ignores things like areas of disks. I do not know what kinds of categorical invariants one can begin to define when we keep track of the Novikov variable.

Remark V.1.0.3. As Kate stated in her talks, there is a result of Abbondandolo and Schwarz that already established some connections between symplectic invariants and string topology: The symplectic homology of T^*M is isomorphic to the homology of the free loop space of M.

Unsurprisingly, this result also has some motivation. It's been known already for a long time that the dynamics of the geodesic flow recover the homology of the free loop space. Indeed, this was one of Morse's original applications – he used his theory to study how the space of geodesics change upon deforming Riemannian metrics. And for people who do rational homotopy theory, the fact that there are such concrete models for the rational homology of loop spaces were the first applications to questions of interest in Riemannian geometry.

Anyhow, the connection of the dynamics of the geodesic flow to symplectic geometry is that the geodesic flow is a Hamiltonian flow for the Hamiltonian called "norm-squared of a covector." This equivalence is an example of the Hamiltonian formulation of the least action principle.

V.2. The basic categories at play

V.2.1. The wrapped Floer cochains of a cotangent fiber. Let Q be a smooth manifold. Then its cotangent bundle T^*Q is a symplectic manifold, and one can define a "wrapped Fukaya category" of T^*Q which studies the symplectic geometry of T^*Q in a way respecting the conical/tubular structure of T^*Q away from the zero section.

Theorem V.2.1.1 (Abouzaid[4].). Let Q be an oriented closed manifold. The[5] wrapped Fukaya category of Q is equivalent to the category of perfect[6] left[7] modules over the A_∞-algebra of chains on the based loop space of Q:

$$\mathcal{W}(T^*Q) \simeq \mathsf{Mod}^{\mathsf{Perf}}(C_*\Omega Q).$$

Remark V.2.1.2. Here is one reason you might believe both sides of this equivalence to have some relation to each other. It turns out that the dynamics of a single cotangent fiber encapsulates everything on the lefthand side – this is Abouzaid's result that the cotangent fiber generates the wrapped Fukaya category. The dynamics can be understood by applying the Hamiltonian flow of a function $|p|$ on parts of the cotangent fiber far away from the zero section; the self-intersections of a cotangent fiber under this flow are lifts of certain closed loops on Q, which one should think of as geodesics – but of particular lengths, and based at the same point the cotangent fiber is based. As we study this dynamic as we increase the slope of $|p|$ to be $k|p|$ for $k \to \infty$, we simultaneously see longer and longer geodesics, and the definition of the wrapped Floer cochains of a cotangent fiber.

On the other hand, I hand-waved earlier that the dynamics of the geodesic flow can recover the homology of the free loop space. One should think of the previous paragraph as a "based loops" version.

Remark V.2.1.3. Abouzaid's theorem about cotangent bundles is evidence that the wrapped Fukaya category – at least this version of it – is a very weak symplectic invariant. After all, it only depends on the homotopy type of Q (together with a choice of orientation and Stiefel-Whitney class of degree 2).

As always, there is a trade-off. The weaker invariants tend to have more structure, and indeed one can prove a lot of interesting things about wrapped Fukaya categories of this "homotopical" flavor.

As I've advertised before, it's okay to have "weaker" symplectic invariants if they give rise to interesting applications in other fields. Indeed, these wrapped Fukaya categories are the first place to try and enrich Fukaya categories over spectra, and hence give rise to ways we might study stable homotopy theory through symplectic geometry.

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[4] Mohammed Abouzaid, "A cotangent fibre generates the Fukaya category." Advances in Mathematics 228 (2011) 894–939

[5] Technically, one must specify a particular structure on the symplectic manifold T^*Q to define orientation data for moduli spaces of J-holomorphic maps, and hence to define any Fukaya-categorical data over $\mathbb{Z}$. Here, that data is canonical – given by the second Stiefel-Whitney class of Q pulled back to T^*Q.

[6] You can think of this as meaning: Finitely generated under shifts and cones from the $C_*\Omega Q$.

[7] The distinction between left and right modules won't matter much for us; because ΩQ is group-like, it is equivalent to its opposite algebra.

V.3. The free loop space via Hochschild homology

Abouzaid's theorem above is the first concrete connection, in our lectures, between Fukaya-categorical ideas and interesting algebra arising from homotopy theory.

And, you might imagine that homotopy theorists know how to compute things about the righthand side of Theorem V.2.1.1 – i.e., about the category of modules over the based loop spaces. It turns out we can. In particular, we can compute the Hochschild homology of this category.

Remark V.3.0.1. We have not talked about Hochschild homology yet; I will introduce it more indepth in Exercise V.10. By definition, the Hochschild homology of a single algebra is the homology of the derived tensor product

$$R \otimes^{\mathbb{L}}_{R \otimes R^{\mathrm{op}}} R$$

of R with itself, over $R \otimes R^{\mathrm{op}}$. (This algebra is a convenient way of encoding R-R bimodule structures.)

There is a definition of bimodules for categories as well, and when a category is generated by a single object, it turns out that the Hochschild homology of the category is equivalent to the Hochschild homology of the endomorphism ring of that single object.

Using this trick, for small-enough categories like Fukaya categories of small-enough symplectic manifolds, one can often reduce computations to the case of a single A_∞-algebra.

Here is the appearance of the free loop space – as a categorically formal construction arising from the based loop space.

Theorem V.3.0.2 (Goodwillie[8], Burghelea-Fiedorowitz[9].). Let Q be a pointed topological space and R any commutative unital ring. Then there exists a natural isomorphism

$$HH_*(C_*(\Omega Q)) \cong H_*(\mathcal{L}Q).$$

That is, the Hochschild homology of chains on the based loop space of Q is isomorphic to the homology of the *free* loop space of Q.

Remark V.3.0.3. In fact, more is true; there is a natural map from one complex to the other that induces this isomorphism on homology – so the two complexes are quasi-isomorphic.

Remark V.3.0.4 (Circle actions). Let's notice that $\mathcal{L}Q$ has a circle action. Indeed, the space of maps $S^1 \to Q$ has a circle action by rotating the domain circle. Kate produced operations on the S^1-equivariant homology of $\mathcal{L}Q$ using this fact.

[8] Goodwillie, T.G. "Cyclic homology, derivations, and the free loopspace." Topology 24.2 (1985): 187-215.

[9] Burghelea, D., Fiedorowicz, Z. "Cyclic homology and algebraic K-theory of spaces—II." Topology 25.3 (1986): 303-317.

It turns out that the lefthand side – Hochschild homology of *any category* – has a circle action as well. This was discovered quite combinatorially/algebraically, and goes back (I believe) to observations by Loday. I think it's fair to say that one of the difficulties in studying this circle action in a geometric setting is that this circle action is so combinatorial; the geometric circle doesn't actually appear until one finagles some things[10], and even then, the appearance doesn't give full access to the circle as a 1-manifold. The language of factorization homology – see III.7 – gives a 1-line proof that there is a natural homotopy action of the oriented diffeomorphism group of the circle on Hochschild chains (and this group is equivalent to the circle group).

Let me outline how this one-line proof goes. It comes down to the observation that Hochschild homology is a left Kan extension. Namely, if $\mathsf{Disk}_1^{\mathrm{or}}$ is the topologically enriched category whose objects are disjoint unions of oriented copies of $\mathbb{R}$ (i.e., open oriented disks) and whose morphisms are smooth, open, orientation-repecting embeddings, then any A_∞ or associative algebra defines a functor

$$\mathsf{Disk}_1^{\mathrm{or}} \to \mathsf{Chain}$$

sending $\coprod_I \mathbb{R}$ to $A^{\otimes I}$. Moreover, it's natural to try to extend such a fuctor to the topologically enriched category of *all* oriented 1-manifolds. There is a categorically formal way of constructing such an extension, called a *left Kan extension*:

$$\begin{tikzcd} \mathsf{Disk}_1^{\mathrm{or}} \arrow[r, "A"] \arrow[d] & \mathsf{Chain} \\ \mathsf{Mfld}_1^{\mathrm{or}}. \arrow[ur, dashed, "\int_\bullet A"'] & \end{tikzcd} \tag{V.3.1}$$

This left Kan extension, by definition, is the factorization homology of A. Moreover, clearly the category $\mathsf{Mfld}_1^{\mathrm{or}}$ has an object called the circle, and the circle has an automorphism space given by the space of orientation-preserving diffeomorphisms of the circle. Noting that $\mathrm{Diff}^+(S^1)$ is equivalent as an $\mathbb{E}_1$-group to S^1 itself, one conclude that $\int_{S^1} A$ has an action by S^1. It is a theorem that $\int_{S^1} A$ – the factorization homology of a circle with coefficient in A – is equivalent to the Hochschild chains of A, and that the circle action observed here is equivalent to the usual circle action on Hochschild chains.

Remark V.3.0.5. Anyway, it was proven in the same work of Goodwillie and of Burghelea-Fiedorowicz that the isomorphism in Theorem V.3.0.2 arises S^1-equivariantly, at the chain level.

By combining the two theorems we have spoken about today, we find the following corollary:

[10] by taking geometric realization of a "cyclic" simplicial object

COROLLARY V.3.0.6. For any oriented smooth compact manifold Q, the Hochschild homology of the wrapped Fukaya category of T^*Q is isomorphic to the homology of the free loop space of Q.

$$HH_*\mathcal{W}(T^*Q) \cong H_*(\mathcal{L}Q).$$

REMARK V.3.0.7. In line with Remark V.3.0.3, again, more is true. Abouzaid's theorem is proven at the chain level, so we in fact have a chain level map from the Hochschild chain complex of $\mathcal{W}(T^*Q)$ to the Hochschild chain complex of (perfect modules over) $C_*\Omega Q$; this in turn admits a quasi-isomorphism to a chain complex computing the homology of the free loop space.

V.3.1. Abbondandolo-Schwarz's result. On the other hand, Kate told us about the following result of Abbondandolo and Schwarz:

THEOREM V.3.1.1 (Abbondandolo-Schwarz[11]). There exists an[12] isomorphism between the "Floer homology" of T^*Q and the homology of the free loop space of Q:

$$H_*(\mathcal{L}Q) \xrightarrow{\cong} HF_*(T^*Q).$$

NOTATION V.3.1.2 (SH). This is a notational and historical note. What Abbondandolo-Schwarz called the "Floer homology" of the cotangent bundle (in analogy with the usual Floer homology of symplectic manifolds, which counts periodic orbits of Hamiltonians) is now called *symplectic homology* of the cotangent bundle.

Thus, instead of following the notation of Abbondandolo-Schwarz, we will follow the notation of symplectic homology. With this meaningless change of notation, the isomorphism from Theorem V.3.1.1 can thus be written as

$$H_*(\mathcal{L}Q) \cong SH_*(T^*Q)$$

where the righthand side is now the notation for symplectic homology.

REMARK V.3.1.3. In analogy with Hamiltonian Floer homology, SH_* is also generated as a chain complex by circles – orbits under a particular sequence of Hamiltonians that look like $k|p|$ for bigger and bigger k. (As another model, one could just take a single Hamiltonian that looks like $|p|^2$; the equivalence of these two models is given by using continuation map arguments based on the fact that $|p|^2$ has bigger derivatives than $k|p|$ eventually.)

[11]Alberto Abbondandolo and Matthias Schwarz. "On the Floer homology of cotangent bundles." Communications on Pure and Applied Mathematics, Volume 59, Issue 2, February 2006. Pages 254-316.

[12]I do not use the word "natural" here; this is because the model that Abbondandolo-Schwarz use does not have the properties that a natural transformation ought to. They use particular Morse-theoretic models.

REMARK V.3.1.4 (Circle actions). As a result, one can imagine creating a model for SH also with a circle action, by keeping track of the parametrization of the generating loops of the complex. As far as I can tell, Abbondandolo-Schwarz do not make contact with circle actions in their paper.

V.3.2. Some conjectures (now proven) inspired by putting it all together. It is never fun to have a million results thrown at you; three theorems aren't a million, but they are still a lot. So let's organize them.

$$\begin{tikzcd}
\mathcal{W}(T^*Q) \arrow[r, "\simeq", "\text{Thm V.2.1.1}"'] \arrow[d, mapsto, "HH_*"] & \mathsf{Mod}^{\mathsf{Perf}}(C_*\Omega Q). \arrow[d, mapsto, "HH_*"] \\
HH_*(\mathcal{W}(T^*Q)) \arrow[r, "\cong"] & HH_*(C_*\Omega Q). \arrow[d, "\cong", "\text{Thm V.3.0.2}"'] \\
& H_*(\mathcal{L}Q) \arrow[d, "\cong", "\text{Thm V.3.1.1}"'] \\
& SH_*(T^*Q)
\end{tikzcd}$$

The first line is an equivalence of A_∞-categories proven by Abouzaid (Theorem V.2.1.1). We obtain the second line by applying Hochschild homology to both sides of the equivalence. The rest of the isomorphisms follow from the theorems referenced.

Of course, this composition tells us that the Hochschild homology of the wrapped category of T^*Q is the symplectic homology of T^*Q. This is a very nice result.

But you should find this proof dissatisfying: The domain and codomain of this isomorphism are both objects that can be articulated using symplectic geometry, yet the isomorphism passes through a purely algebraic result. Is there a more geometric way to see this? In other words, re-writing the bottom of the above diagram, is there a dashed map making the following diagram commute?

$$\begin{tikzcd}
HH_*(\mathcal{W}(T^*Q)) \arrow[r, "\cong"] \arrow[d, dashed] & HH_*(C_*\Omega Q). \arrow[d, "\cong", "\text{Thm V.3.0.2}"'] \\
SH_*(T^*Q) \arrow[r, "\cong", "\text{Thm V.3.1.1}"'] & H_*(\mathcal{L}Q)
\end{tikzcd}$$

Moreover, the righthand vertical arrow is known to be equivariant with respect to the circle action on both domain and codomain (Remark V.3.0.5). Thus, could one make a geometric map in the lefthand side that also respects a circle action?

We may talk more about this later; let me just say that the most natural constructions I've seen have to do with HH^* – Hochschild *co*homology – and not HH_*. We'll talk about HH^* next.

Remark V.3.2.1. Wanting a more geometric model for this isomorphism isn't just a matter of mathematical aesthetics. In general – as we've seen in studying both operads and in setting up the Fukaya category – the geometry is the source of algebraic structures, so one can imagine that to prove results in the future, one wants results that are geometric in nature. The closer the results are to the starting points, the easier it is to prove things about the starting points to prove more universal results.

And, the dashed arrow in the above diagram is the start of a trend in this lecture. A ton of conclusions about geometric actions will only follow from categorical results; I don't think many people even want to bother writing down purely geometric proofs of some of these structures directly from the geometric definitions.

V.4. (Not covered in lecture) The appearance of $\mathbb{E}_2$: Dunn additivity and Hochschild cohomolgy

The one-line conclusion of this section is that "Hochschild cochains form an $\mathbb{E}_2$-algebra." One would conjecture this to be true if one knew about Dunn additivity.

V.4.1. To be $\mathbb{E}_2$ is to be $\mathbb{E}_1$ in $\mathbb{E}_1$. The following theorem is a key tool for understanding and detecting examples of $\mathbb{E}_n$-operads:

Theorem V.4.1.1 (Dunn additivity). *The ∞-category of $\mathbb{E}_{n+1}$-algebras is equivalent to the ∞-category of $\mathbb{E}_1$-algebras in the ∞-category of $\mathbb{E}_n$-algebras.*

This theorem, in its original guise, was proven by Gerald Dunn; the paper was published in 1988.[13] The paper's results translate into the formulation above.

Example V.4.1.2. Suppose that A is an $\mathbb{E}_1$-algebra, meaning in particular that it is equipped with binary operations that are associative up to homotopy.

Suppose that you give A another structure of an $\mathbb{E}_1$-algebra, and that in fact, this is a structure of A as an $\mathbb{E}_1$ algebra *of* $\mathbb{E}_1$*-algebras*.

(The classical analogue would be to give A an associative algebra structure for which the multiplication map $m : A \otimes A \to A$ is a map of associative algebras.)

Then, by Dunn additivity, there exists a canonical structure of an $\mathbb{E}_2$-algebra on A.

[13]Gerald Dunn, "Tensor product of operads and iterated loop spaces," Journal of Pure and Applied Algebra, Volume 50, Issue 3, 1988, Pages 237-258, ISSN 0022-4049, `https://doi.org/10.1016/0022-4049(88)90103-X`.

Remark V.4.1.3. Informally, the Dunn Additivity theorem states that if you can give an object n compatible multiplications, then the object is an $\mathbb{E}_n$-algebra. This statement is most interesting in a setting where there are homotopical structures present. In the exercises – Exercise IV.13 – you've seen that $\mathbb{E}_n$ for $n \geq 2$ is the same thing as being commutative (on the nose) in the setting of abelian groups, or of sets.

In fact, more is true: If you can give an algebra n compatible multiplications, they are all homotopic to each other.

V.4.2. Hochschild cochains are an $\mathbb{E}_2$-algebra. So, what are Hochschild cochains?

Definition V.4.2.1 (Hochschild cochains, informal). Fix a category $\mathcal{C}$. Note that the identity $\mathrm{id}_{\mathcal{C}}$ is a functor from $\mathcal{C}$ to itself. we define the *Hochschild cochain complex* of $\mathcal{C}$ to be the endomorphisms of $\mathrm{id}_{\mathcal{C}}$ – that is, the collection of natural transformations from $\mathrm{id}_{\mathcal{C}}$ to itself.

Remark V.4.2.2. As suggested by the name Hochschild *cochains*, this definition is most often applied when $\mathcal{C}$ is a dg- or A_∞-category. In such a setting, indeed the collection of natural transformations forms a cochain complex.

Remark V.4.2.3. The definition above is informal; we will give a more concrete model for it in Exercise V.11 when our category $\mathcal{C}$ has only one object, and whose endomorphism ring is a ring R concentrated in degree 0. In this setting, one computes the "derived endomorphisms" of the identity functor to compute Hochschild cochains.

For now, let's give some intuition as to why something like Hochschild cochains should have an $\mathbb{E}_2$-algebra structure. We will rely on Dunn additivity (Theorem V.4.1.1 and Remark V.4.1.3). Recall that the Dunn Additivity theorem tells us that if a single object can be given two compatible multiplications, then that object can be given the structure of an $\mathbb{E}_2$-algebra.

The collection of natural transformations $f : \mathrm{id} \to \mathrm{id}$ has exactly this structure. Because id is an idempotent functor, there is a natural equivalence $\mathrm{id} \circ \mathrm{id} \simeq \mathrm{id}$ as functors. This endows the collection of natural endomorphisms $a : \mathrm{id} \to \mathrm{id}$ with two different compositions: One obtained by just composing the various a, and another given by post- or pre-composing a by the operation of multiplying id with itself. (This takes advantage of how categories, functors, and natural transformations behave like a 2-category – see Exercise V.9.)

Thus, one is led to conjecture that Hochschild cochains form an $\mathbb{E}_2$-algebra. This is true in great generality, and the original proof goes back to Tamarkin, who proved it in the dg-setting.

Theorem V.4.2.4 (The Deligne Conjecture). Let $\mathcal{C}$ be a dg-category, or more generally, an A_∞-category. Then the Hochschild cochain complex of $\mathcal{C}$ has the structure of an $\mathbb{E}_2$-algebra in chain complexes.

Remark V.4.2.5. Confusingly, this is one of those conjectures which – after its affirmative resolution – did not change its name after becoming a theorem.

Remark V.4.2.6. As far as I know, Deligne did not conjecture the theorem via Dunn Additivity (even though Dunn Additivity was published already). Instead, in a letter in 1993, Deligne noted that the Hochschild cohomology groups of an algebra had the structure of a Gerstenhaber algebra (i.e., an algebra over the cohomology of the $\mathbb{E}_2$-operad) and wondered if this cohomology level action lifted to the chain level. I say "as far as I know" because I have never seen the original letter of Deligne.

Remark V.4.2.7. By the way, Gerstenhaber[14] first computed that HH^* has a commutative product and a degree 1 Poisson bracket in the 1960's. This is the origin of the term "Gerstenhaber algebra." Then, in the 1970's, Fred Cohen[15] showed that any algebra over the $\mathbb{E}_2$-operad has the property that its homology is a Gerstenhaber algebra (though he did not use that term). It took nearly twenty years until Deligne, in 1993, noted the connection in a letter to Stasheff and others. It is fun to daydream about what interesting structures today will not have some obvious connections noticed for 20 years.

V.4.3. Hochschild cohomology is a Gerstenhaber algebra. Now that we know that Hochschild cochains of an A_∞-category are an $\mathbb{E}_2$-algebra, let's simplify things by taking homology of the complexes. Then the homology of (the spaces in the) $\mathbb{E}_2$ operad acts on the homology of the Hochschild complex. More precisely, there is a new operad (in graded abelian groups) given by homology of $\mathbb{E}_2$, and it acts on the graded abelian group called HH^*.

Proposition V.4.3.1. If a graded abelian group A is an algebra for the homology of the $\mathbb{E}_2$ operad, then the graded abelian group inherits

(a) A graded-commutative product $A \otimes A \to A$.[16] This renders A a unital, graded-commutative ring.
(b) A Lie bracket of degree 1. More precisely, there is a bilinear map $A \otimes A \to A[1]$ which is graded skew-commutative:
$$\{a, b\} = (-1)^{|a||b|}\{b, a\}.$$
This renders $A[1]$ a graded Lie algebra with bracket degree 0.

[14] Gerstenhaber, M. The cohomology structure of an associative ring. Annals of Math. 78 (1963), 267–288.

[15] Cohen, F.R., Lada, T.J., and May, J.P. The homology of iterated loop spaces. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[16] This comes from $H_0\mathbb{E}_2(2) \cong k$; that this is graded-commutative on the nose is for the same reason that an $\mathbb{E}_2$ multiplication is commutative up to homotopy.

Moreover, these two structures are compatible in that the bracket acts as a graded derivation of degree 1 on the product:

$$\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|+1)} b\{a, c\}.$$

Remark V.4.3.2. A Poisson algebra (of degree 0 – i.e., in the classical sense) is a commutative ring together with a Lie algebra structure where the bracket acts as a derivation on the mulitiplication.

Thus, you should think of the algebraic structure in Proposition V.4.3.1 as like a Poisson algebra structure, but where the bracket has been shifted by 1 degree.

Remark V.4.3.3. If you are not comfortable with all the exponents of -1 showing up in graded formulas, check out Exercise IV.11 where you'll get practice with the Koszul sign rule.

Any graded abelian group with the above structures is called a *Gerstenhaber algebra*. What we see is that HH^* is a Gerstenhaber algebra.

V.5. Wishlists from string topology

The upshot of the previous section is that Hochschild cohomology of any dg- or A_∞-category has a Gerstenhaber algebra structure on it. Now, if there were – in addition to the above structures – a degree 1 , one-input operation corresponding to a circle action, we would call A a BV algebra[17].

We saw from Kate's talks that $H_*(\mathcal{L}M)$, with a shift, had a structure that was exactly that of a BV algebra. Let's organize the structures we've seen in an ad hoc table. The tables don't match up in an apparently clean way at the moment:

Categorical	String topology
Hochschild *homology* has a circle action.	The homology of a free loop space has a circle action.
Hochschild *co*homology has a Gerstenhaber algebra structure.	The homology of a free loop space has a *shifted* Gerstenhaber algebra structure, somehow compatible with the above circle action.

Remark V.5.0.1 (Naive thought 1). Things would match up far more nicely if there were a way to relate Hochschild homology and Hochschild cohomology to each other, with a shift.

And, there is more. Let's assume that you don't care about the above mismatch, but you are interested in proving some of the claims/conjectures we saw in Kate's talks. For example, is it possible that the moduli of Riemann surfaces acts on the homology of the free loop space?

[17] I am not spelling out here the compatibility between this circle operator and the other structures of bracket and product; see Kate's notes/exercises.

You've probably seen by now that there is a helpful philosophy in this higher-algebra game. Just as with the A_∞-relations in Fukaya categories, we should look for universal explanations for algebraic structures by simply understanding the correct moduli spaces. In your situation, is there some single system of geometric moduli spaces that controls all the structure that you want to encode?

When we squinted yesterday, we saw that an A_∞-category is the same thing as a structure living over the moduli space of marked holomorphic disks, with an extra detail – we cared about which marked point was the 0th marked point, so we could talk about outputs of composition. Imagine for a moment that we have a structure that is just like an A_∞-category, but with a way of not caring about the linear order (e.g., 0th point) of boundary marked points of disks.

$$\text{Disks w } \partial \text{ marked points} \xrightarrow{\text{something cyclic on an } A_\infty\text{-cat}} \mathsf{Chain}$$

Then, the moduli of marked holomorphic disks is just a small part of a bigger moduli space – the moduli space of Riemann surfaces (with marked points on the boundary). And category theory has formal techniques – like Kan extension – that can extend the above structure:

(V.5.1)

$$\begin{tikzcd}
\text{Disks w } \partial \text{ marked points} \arrow[r, "\text{something cyclic on an } A_\infty\text{-cat}"] \arrow[d] & \mathsf{Chain} \\
\text{Riemman surfaces w } \partial \text{ marked points} \arrow[ur, dashed, "\text{Kan extension}"'] &
\end{tikzcd}$$

This should be compared, of course, to (V.3.1).

Remark V.5.0.2 (Naive thought 2). What if there was a notion of A_∞-category with a "cyclic" structure like the one imagined above? Does it have categorical invariants with actions of the moduli of all Riemann surfaces?

V.6. Calabi-Yau structures and Costello's theorem

Amazingly, both naive thoughts above have a simultaneous and a clever observation – I am not sure to whom it is due, but I learned it from Costello's paper[18], and it may admit some predecessor's in talks of Kontsevich – bootstrapping off the observation we made last lecture about A_∞-relations and moduli of disks with marked points.

It turns out that a very popular and naturally-occurring categorical structure that gives rise to such a structure is that of a *Calabi-Yau* category.

Definition V.6.0.1 (Calabi-Yau structure of dimension d). Let $\mathcal{C}$ be a dg- or A_∞-category. A *Calabi-Yau structure* of dimension d on $\mathcal{C}$ is the data

[18] Kevin Costello, "Topological conformal field theories and Calabi–Yau categories." Advances in Mathematics 210 (2007) 165–214.

of a cyclically invariant pairing on $\mathcal{C}$ that is non-degenerate with a shift of degree d.

A more concrete definition can be found in, for example, the famous paper of Costello – see Exercise V.12. For now, let me say that this pairing is extra data. It in particular supplies, for every ordered pair of objects of $\mathcal{C}$, a linear map

$$\langle , \rangle_{X,Y} : \hom(X,Y) \otimes \hom(Y,X) \to R[d]$$

where R is my base ring. Such a pairing is called symmetric if

$$\langle , \rangle_{X,Y} = \langle , \rangle_{Y,X}$$

and non-degenerate if the induced map

$$\hom(X,Y) \to \hom(Y,X)^\vee[d] = \hom_{\mathsf{Chain}}(\hom(Y,X), R[d])$$

to the dual is an equivalence.

Example V.6.0.2. For a dg- or A_∞-category to admit a Calabi-Yau structure is a very restrictive condition. For example, if X and Y are objects with no negative homs between them in either direction[19], the hom complexes looks like

$$\ldots \to 0 \to \hom^0(X,Y) \to \hom^1(X,Y) \to \hom^2(X,Y) \to \ldots$$

and

$$\ldots \to 0 \to \hom^0(Y,X) \to \hom^1(Y,X) \to \hom^2(Y,X) \to \ldots.$$

In particular, the dual complex of the latter is concentrated in non-positive degrees:

$$\ldots \to (\hom^2(Y,X))^\vee \to (\hom^1(Y,X))^\vee \to (\hom^0(Y,X))^\vee \to 0$$

So for the $\hom(X,Y)$ chain complex to be equivalent to a finite d-shift of $\hom(Y,X)^\vee$, we see that both $\hom(Y,X)$ and $\hom(X,Y)$ must have no cohomology above degree d.

This constraint on the homological range in which hom complexes can exist uses only the non-degeneracy condition of the pairing.

Remark V.6.0.3. A consequence of being Calabi-Yau is an isomorphism

$$HH_i(\mathcal{C}) \cong HH^{d+i}(\mathcal{C})^\vee$$

between Hochschild homology and the linear dual of Hochschild cohomology, up to a shift of degree d.

The connection to the moduli of Riemann surfaces is as follows:

[19] for example, if X and Y are both vector bundles and if your category is a dg-version of D^bCoh

THEOREM V.6.0.4 (Costello[20]; Hiro will leave things vague.). Let $\mathcal{C}$ be a Calabi-Yau dg- or A_∞-category. Then the Hochschild chain complex of $\mathcal{C}$ has an action by the PROP of chains on the moduli of Riemann surfaces.

REMARK V.6.0.5. This comes very close to fulfilling Kate's wish of having the moduli of Riemann surfaces act on $H_*(\mathcal{L}M)$, but the theorem cannot accomplish this. The reason Costello's result cannot produce this is that the hom-complexes of the relevant categories are too large – neither $\mathcal{W}(T^*Q)$, nor modules over chains on the based loop space, have finitely concentrated hom complexes. On the other hand, the recent pre-print of Kontsevich-Takeda-Vlassopoulos does give what Kate wants.

V.7. An outline of the proof of Costello's theorem

The proof is remarkable and beautiful; the introduction of the paper is worth reading. Indeed, Lurie's paper on the Cobordism Hypothesis cites Costello's work as the inspiration.

V.7.1. A_∞-categories are certain functors out of a category with disk-morphisms with one output. First, Costello fixes a collection Λ of objects (think of them as Lagrangians in a symplectic manifold) and defines a category whose morphisms are holomorphic disks with marked points on the boundary where exactly one is marked as outgoing, and the others are marked as incoming; each morphism also has labels by elements of Λ.

This is confusing, so read carefully: One thinks of each ordered pair (λ, λ'), and more generally, each tuple of ordered pairs

$$\big((\lambda_i, \lambda_i')\big)_i$$

as an object. Between any two such tuples of pairs, one can construct a moduli space of Riemann surfaces with boundary, with boundary arcs given labels given by the λ_i, and labels interrupted by marked points. By taking chains on these moduli of Riemann surfaces, we witness a category enriched in chain complexes. We'll let $\mathcal{D}^+$ denote the subcategory consisting only of those Riemann surfaces that are nodal disks with exactly one output marked point (and ≥ 0 input marked points). Then a functor

$$\mathcal{D}^+ \to \mathsf{Chain}_R$$

sending $\coprod$ of labels to $\otimes$, is precisely the data of an A_∞-category. Concretely, to a pair λ, λ' one associates a chain complex $V_{\lambda,\lambda'}$. To a disk with k inputs and one output, one thus assigns a linear map $V^{\otimes k} \to V$. The moduli of such disks has dimension $k-2$ as we discussed last time, so one can think of this as a single linear map in degree $k-2$.

[20] Kevin Costello, "Topological conformal field theories and Calabi–Yau categories." Advances in Mathematics 210 (2007) 165–214.

Remark V.7.1.1 (A description of $\mathcal{D}^+$). There are in fact some unnatural things about Costello's construction that I have not yet fully understood, except to see that they are necessary for the conclusions. The disk with exactly one marked output point (and no input points) is formally declared to have a 0-dimensional moduli space called a point, and it is meant to pick out the identity map $R \to V_{\lambda,\lambda}$. The disk with exactly one input and one output marked point (a strip) is required to pick out the identity map $V_{\lambda,\lambda'} \to V_{\lambda,\lambda'}$ of hom complexes. Indeed, this follows from the definition of composition in Costello's category $\mathcal{D}^+$: Strips glue onto other disks without changing those disks, while the 0-input disk glues with the effect of eliminating a marked input point. All other disks "compose" by concatenating along nodes, producing Mickey Mouse pictures.

Pictures.

Remark V.7.1.2. The observation that certain functor outs of $\mathcal{D}^+$ is the same thing as an A_∞-category is in fact quite useful. By noting that $\mathcal{D}^+$ is formal, Costello actually proves in this paper that the theory of A_∞-categories is equivalent to the theory of dg-categories. This proof is very different from the usual proof passing through some form of Yoneda (for example, the proof sketched in Chapter One of Seidel's book).

V.7.2. Calabi-Yau categories are functors out of a cyclic nodal disk category. Now let

$$\mathcal{D}_{open}$$

denote the category with the same objects, but with more morphisms: Now disks are allowed to have arbitrarily many inputs and outputs, subject to a Zorro's Lemma relation: Concatenating a 2-input-0-output disk with a 2-output-2-input disk along one node results in a 1-input-1-output strip (which, by our discussion of $\mathcal{D}_+$, acts as the identity under concatenation).

Pictures.

Moreover, we place a shifted local system on the moduli of such disks; this has the effect that any functor

$$\mathcal{D}_{open} \to \mathsf{Chain}_R \tag{V.7.1}$$

will take a disk with two input punctures to a degree d pairing

$$V_{\lambda,\lambda'} \otimes V_{\lambda',\lambda} \to R[d].$$

The Zorro relation referred to above renders this pairing non-degenerate. The fact that disks have cyclic symmetries renders this pairing cyclically symmetric. In other words, a functor as in (V.7.1) is precisely the data of a Calabi-Yau category.

V.7.3. The proof of Costello's theorem. Now, Costello notes that $\mathcal{D}_{open}$ naturally sits inside a category $\mathcal{OC}$ of "open closed" Riemann surfaces as morphisms. In particular, this category has more objects given by boundary circles of Riemann surfaces with no marked points. Given a Calabi-Yau

A_∞-category $\mathcal{C}$, one can formally compute the left Kan extension, realizing the diagram from the wishlist (V.5.1):

$$\begin{tikzcd}
\mathcal{D}_{open} \arrow[r, "{\mathcal{C},\langle,\rangle}"] \arrow[d] & \mathsf{Chain}_R \\
\mathcal{OC} \arrow[ur, "\text{left Kan extension}"'] &
\end{tikzcd}$$

The remarkable theorem is that this left Kan extension evaluates on the circle exactly the Hochschild chain complex of $\mathcal{C}$. In particular, for free, the moduli of all Riemann surfaces acts on Hochschild chains, and in fact, in an open-closed way (e.g., receiving actions from the homs of the ∞-category $\mathcal{C}$ as well).

Remark V.7.3.1. The proof of the theorem ultimately relies on a careful construction, by Costello, of a particular "cellular" decomposition for the moduli of Riemann surfaces.

Remark V.7.3.2. The remarkable thing about this proof strategy is that the action is exhibited by a universal property. Indeed, by definition of the left Kan extension, this action is initial among all reasonable actions one could write. This is a far stronger result than just writing an action by hand.

Remark V.7.3.3. The above proof should of course be compared to (V.3.1). Factorization homology is defined in a completely analogous way: One begins with a basic structure (such as an $\mathbb{E}_n$-algebra), realizes the structure is equivalent to a functor out of some geometric ∞-category (of n-dimensional framed disks) and left Kan extends to a more interesting ∞-category (of all n-dimensional framed manifolds).

V.8. pre-Calabi-Yau structures

Now let me try to give a rough idea of what Kontsevich-Takeda-Vlassopoulos do. I have not read their proofs, though, so my discussion here will be quite superficial. It also seems that various versions of the pre-print had already been circulating among experts, but I had never seen it, so I was quite excited to skim the paper.

Here are three big steps forward taken in the preprint:

(a) They introduce the notion of a *pre-Calabi-Yau* structure of dimension d. As you saw above, the notion of being Calabi-Yau imposed very strong conditions on how finite hom-complexes could be (and another subtle homological condition on whether the category itself is smooth; that's another topic). A pre-Calabi-Yau structure can exist on categories without such strong homological constraints. Moreover, a pre-Calabi-Yau structure has beautiful geometric interpretations in non-commutative geometry (Remark V.8.0.2).

(b) They create a model of the moduli of Riemann surfaces that is, apparently, quite useful. Indeed, the preprint cites prior work of Tradler, Zeinalian, Poirier; but it seems the big insight is the organizational power of certain quadratic differentials (familiar from the theory of flat surfaces).

(c) Combining the two, the authors prove that for any pre-Calabi-Yau A_∞-category, the Hochschild chains has an action by the PROP of chains on the moduli of Riemann surfaces, where the chains are twisted by a local system shifted in degrees depending on d.

It's a result of Ralph Cohen and Sheel Ganatra[21] that the wrapped Fukaya category of T^*Q, and chains on the based loop space, have compatible pre-Calabi-Yau structures (in fact, an even more restrictive structure of what's called "left Calabi-Yau"). Combining the three-author pre-print with the based-loop-space side of the result of Cohen-Ganatra, we find:

Theorem V.8.0.1. For any oriented, smooth, compact manifold M, chains on the free loop space $C_*\Omega M$ enjoy an action from the PROP of chains (with coefficients in a shifted local system) on the moduli of Riemann surfaces.

Remark V.8.0.2. Because Kate stated as a conjecture the action of this PROP on chains on the based loop space, it seemed climactic to mention the result of Kontsevich-Takeda-Vlassopoulous.

But I want to talk a little bit about why the definition of pre-Calabi-Yau structure is exciting.

If you're a symplectic geometer, you probably know the difference between a symplectic and a Poisson structure. The former is controlled, well-behaved; the latter is much more wild. But from the perspective of quantization, it's a Poisson structure that gives you a way to quantize the commutative structure of C^∞ functions on a manifold.

It turns out that a pre-Calabi-Yau structure is a derived, non-commutative geometry generalization; that is, if I understand correctly, the three authors conjecture that a pre-Calabi-Yauness on an A_∞-algebra is exactly the algebraic analogue of placing a Poisson structure on a manifold.

A concrete manifestation of this claim would be to witness a shifted Poisson structure on the moduli of representations of the given A_∞-algebra.

There is yet another mysterious interpretation. Using Koszul duality, one can think of an A_∞-structure on a graded vector space as the same thing as giving a degree 1 vector field (a coderivation) on a formal "graded manifold" (the free coalgebra on the graded vector space). This amounts to writing down a solution to the Maurer-Cartan equation in the Lie algebra structure of the 1-output Hochschild cochain complex $\hom(V^{\otimes k}, V)$. Apparently, there is a l-output version one can write down for all $l \geq 1$ at once,

[21]This is summarized in a pre-print available on Ralph Cohen's website. "Calabi-Yau categories, the Floer theory of a cotangent bundle, and the string topology of the base."

again with a Lie bracket, and a solution to the Maurer-Cartan equation seems to be the data of a bunch of polyvector fields that are involutive; i.e., the analogue of giving a foliating polyvector field.

I do not know if these foliations are non-commutative generalizations of the foliation theory that Toën and Vezzosi are developing in the setting of derived algebraic geometry.[22]

[22]See the pre-prints Bertrand Toën, Gabriele Vezzosi. "Foliations and stable maps," arXiv:2202.09174 and Bertrand Toën, " Classes caractéristiques des schémas feuilletés " arXiv:2008.10489.

Exercises

V.9. The 2-categorical structure of categories

Fix three categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, and six functors

$$F_i, G_i, H_i : \mathcal{C}_i \to \mathcal{C}_{i+1}, \qquad i = 1, 2.$$

Also fix natural transformations $a : F_1 \to G_1$ and $b_1 : G_1 \to H_1$.

(a) Convince yourself that the composition $b_1 \circ a_1$ makes sense, and is still a natural transformation.
(b) Convince yourself that the notation $G_2 \circ a_1$ makes sense – as does $H_2 \circ b_1$.
(c) Write out the ways in which you can obtain a single natural transformation from $F_2 \circ F_1$ to $H_2 \circ H_1$. Are there "two notions of composition" you are using? How are they compatible?
(d) Suppose now that $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3$ and $F = G = H$, and you are supplied with a natural isomorphism $F \circ F \xrightarrow{\cong} F$. Can you prove that the collection of natural transformations of F is a commutative monoid?

V.10. Hochschild homology

Throughout, we fix an associative unital ring R over a base ring k, and we let $\otimes = \otimes_k$. For simplicity, everything below assumes that R is concentrated in degree 0 – i.e., is an ordinary ring, and not a cdga or a dga.

(a) Let R^{op} be the opposite algebra. Concretely, the multiplication in R^{op} is given by

$$x *_{\mathrm{op}} y := y * x$$

where $*$ is the original multiplication in R.
(b) Let M be a k-module. Show that the structure of an R-R-bimodule on M is equivalent to given an $R \otimes_k R^{\mathrm{op}}$ left module structure on M. More concretely, exhibit an equivalence of k-linear categories between the category of R-R bimodules and the category of left $R \otimes_k R^{\mathrm{op}}$ modules.

Definition V.10.0.1. Let M be an R-R bimodule. Then the nth *Hochschild homology of R with coefficients in M* is the tor group

$$HH_n(R; M) := \mathrm{Tor}_n^{R \otimes R^{\mathrm{op}}}(R, M)$$

where R is considered a bimodule over itself. When $R = M$, we simply say the above tor groups are the Hochschild homology of R. In this case, we use the notation

$$HH_n(R).$$

Warning V.10.0.2. There are particular chain complexes that compute the Hochschild homology groups. Depending on the circle of math you're in, you may hear mathematicians saying "Hochschild homology" to actually mean this chain complex (even though "Hochschild chain complex" is also a term for this exact thing). In which case, they would define Hochschild homology to be the following chain complex:

$$R \otimes^{\mathbb{L}}_{R \otimes R^{\mathrm{op}}} M.$$

– i.e., as the derived tensor product of R with M over $R \otimes R^{\mathrm{op}}$.

This gives an indication of how would defines Hochschild homology when R is more generally a ring spectrum. Given a bimodule M, one defines the spectrum

$$R \otimes_{R \otimes R^{\mathrm{op}}} M$$

be the "Hochschild homology" spectrum of R with coefficients in M.

(c) By using the fact that Tor_0 computes the usual tensor product, exhibit an isomorphism

$$HH_0(R) = R/[R, R].$$

Remark V.10.0.3. That is, $HH_0(R)$ looks like the universal k-module classifying maps t out of R satisfying[23] $t(xy) = t(yx)$. Such t are called *traces* on R.

Thus, you should think of the Hochschild chain complex of R as, more generally, the derived object classifying all derived traces out of R.

(d) When $R = M$ and R is commutative (and if you are familiar with algebraic geometry) explain why the Hochschild chain complex of R can be interpreted as computing (the space of functions on) a "derived" self-intersection of the diagonal of $\operatorname{Spec} R \times \operatorname{Spec} R = \operatorname{Spec}(R \otimes R)$.

Remark V.10.0.4. Intuitively, a derived intersection is supposed to keep track of the set-theoretic intersection, together with data on how one can deform the intersection if one wanted to. When R is a commutative ring encoding a nice, smooth object, one might then imagine that the derived intersection of R has something to do with the normal bundle of $\operatorname{Spec} R$ inside $\operatorname{Spec} R \times \operatorname{Spec} R$, hence the tangent bundle. You can look up the Hochschild-Kostant-Rosenberg theorem, which computes $HH_*(R)$ in nice examples – indeed, $HH_*(R)$ computes the group of deRham algebraic forms (i.e., the space of differential forms on $\operatorname{Spec} R$). This is quite natural if you believe this derived intersection should have something to do with the tangent bundle – $HH_*(R)$ is like a polynomial algebra generated by functions on the tangent bundle. (Every tangent

[23]By the way, a common mistake that linear algebra students make is to believe this equation implies that "$t(xyz) = t(xzy)$". This quoted equation is false. For example, verify that the usual trace of n-by-n matrices is only invariant under cyclic permutations of matrix products, but not by arbitrary permutations.

vector field on $\operatorname{Spec} R$ – aka a derivation on R – gives rise to a number by the universal property of algebraic Kahler differentials.)

This is a wonderful result, and a funny one: The homology of something computes a complex which is supposed to have a natural differential. (That is, the homology groups of Hochschild chains computes the deRham forms themselves, not the deRham cohomology.) It turns out that Hochschild homology has a degree ± 1 operator arising from a circle action on Hochschild chains, and this operator recovers precisely the deRham differential.

Notice also something funny. The ith Hochschild *homology* of R computes the ith wedge powers of deRham forms (which one normally thinks of as in degree i with *cohomological* conventions).

Remark V.10.0.5. For a less commutative example, consider $C_*\Omega X$, chains on the based loop space of X. Because ΩX is an $\mathbb{E}_1$-space, chains on it is an $\mathbb{E}_1$-algebra, and it makes sense to speak of the Hochschild chain complex on such an "associative" algebra. The theorem of Goodwillie is that

$$HH_*(C_*(\Omega X)) \cong H_*\mathcal{L}X.$$

That is, Hochschild homology of chains on the based loop space is the homology of the free loop space of X. For some reason, the more popular citation of this theorem requires that X be simply-connected, but that is not at all needed. (This is probably a confusion of a result of Goodwillie with a result of Jones.) The only assumption needed is that X is connected.

(e) Look up the definition of the Hochschild chain complex. It looks like a chain complex generated by a simplicial object with terms

$$\dots R^{\otimes n} \otimes M \xrightarrow{n+1} \dots R \otimes R \otimes M \xrightarrow{3} R \otimes M \xrightarrow{2} M.$$

Here, the superscripts over the arrows indicate how many simplicial face maps there are; so M is the 0th level of the simplicial set, and there are two boundary maps from $R \otimes M$ to M. These maps act by taking

$$r \otimes m \mapsto rm \pm mr$$

(and hence uses the bimodule structure on M).

Remark V.10.0.6. The Hochschild chain complex is a very nice presentation of the derived tensor product $R \otimes_{R \otimes R^{\mathrm{op}}} M$, but – just like the bar resolution, or like singular chain complexes of a space – is used mainly to think about functoriality properties of Hcohschild chains. People who know how to do computations would rarely work straight from the definition of the Hochschild chain complex to begin a computation.

(f) When $M = R$, what does $HH_1(R)$ have to do with derivations on R? (Hint: Given a derivation from R to R, does one obtain a number from an element of $HH_1(R)$?)

V.11. Hocschild cohomology

(a) Let $\mathcal{C}, \mathcal{D}$ be two categories and fix two functors $F, G : \mathcal{C} \to \mathcal{D}$. Recall that a *natural transformation* from F to G is a choice of morphisms

$$\eta_c : Fc \to Gc, \qquad c \in \operatorname{Ob} \mathcal{C}$$

such that the diagram

$$\begin{tikzcd} Fc \arrow[r] \arrow[d] & Gc \arrow[d] \\ Fc' \arrow[r] & Gc' \end{tikzcd}$$

commutes for any morphism $c \to c'$ in $\mathcal{C}$.

(b) Let $\mathcal{C}$ be the category of left R-modules. Then there is an endofunctor of $\mathcal{C}$ called the identity functor. Identify the set of natural transformations of $\mathrm{id}_{\mathcal{C}}$ with itself with the center of R.

Definition V.11.0.1. Let M be an R-R bimodule. Then the nth *Hochschild cohomology of R with coefficients in M* is the Ext group

$$HH^n(R; M) := \operatorname{Ext}^n_{R \otimes R^{\mathrm{op}}}(R, M)$$

where R is considered a bimodule over itself. When $R = M$, we simply say the above groups are the Hochschild cohomology of R. In this case, we use the notation

$$HH^n(R).$$

(c) Look up the *Hochchild cochain complex*. It is a complex whose cohomology computes Hochschild cohomology. It looks like

$$\ldots \to \hom_k(R^{\otimes n}, M) \to \ldots \to \hom_k(R^{\otimes 2}, M) \to \hom_k(R, M) \to \hom_k(k, M).$$

Given $f \in \hom_k(R^{\otimes n}, M)$ in degree n, the differential is given by

$$\begin{aligned} (df)(a_1, \ldots, a_{n+1}) = {} & \pm a_1 f(a_2, \ldots, a_{n+1}) \\ & + \sum \pm f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_{n+1}) \\ & \pm f(a_1 \otimes \ldots \otimes a_n) a_{n+1}. \end{aligned}$$

I know the signs above are vague, but once you look up the correct signs, show that $HH^0(R)$ is the center of R.

(d) A version of the HKR theorem from the previous exercises states that, when R is commutative and nice enough, $HH^n(R)$ is the collection of n-polyvector fields on $\operatorname{Spec} R$. Specifically,
- $HH^0(R) = R$, interpreted as the collection of functions on $\operatorname{Spec} R$.
- $HH^1(R) = \Gamma(T \operatorname{Spec} R)$, the collection of vector fields on $\operatorname{Spec} R$; this is also known as the collection of derivations from R to itself.
- $HH^n(R)$ is the nth wedge power (over R) of $HH^1(R)$.

Interpreting, then, the degree 1 part of $HH^*(R)$ as the space of vector fields on something, why would you expect $HH^1(R)$ to have a "Lie bracket of degree 1?" (Recall that the space of vector fields usually has a Lie bracket (of degree 0); and that's when that space is in degree 0.)

(e) Inspired by the above analogy, look up the Schouten brakcet, or the Gerstenhaber bracket, for Hochschild cohomology. Recalling that a vector field on a manifold is the same thing as a smooth derivation, does this bracket look the way you would expect?

(f) Coming back to the idea that Hochschild cochains is an $\mathbb{E}_2$ algebra, we know that HH^* ought to be a Gerstenhaber algebra. Is the space of polyvector fields a Gerstenhaber algebra?

Exercises on Calabi-Yau categories

V.12. One definition of being Calabi-Yau

The following definition can be found in Costello's paper[24] We fix a base ring R.

Definition V.12.0.1. A *Calabi-Yau structure* of dimension d on an A_∞-category $\mathcal{C}$ is the data of – for every object $A, B \in \mathcal{C}$ – a map of chain complexes

$$\hom(A, B) \otimes \hom(B, A) \to R[d]$$

which is non-degenerate, satisfying the following two conditions:

(i) (The pairing is symmetric.) Call the above pairing $\langle , \rangle_{A,B}$. Then

$$\langle , \rangle_{A,B} = \langle , \rangle_{B,A} \circ \mathrm{swap}$$

where swap is the swap $\hom(A, B) \otimes \hom(B, A) \cong \hom(B, A) \otimes \hom(A, B)$ of chain complexes. (This swap does not use any structure on $\mathcal{C}$.)

(ii) More generally, we demand that the pairing be cyclically invariant, meaning

$$\langle m_n(a_0 \otimes \ldots \otimes a_{n-1}, a_n\rangle = \pm\langle m_n(a_1 \otimes \ldots \otimes a_{n-1}, a_0\rangle$$

(a) Show that if $\mathcal{C}$ has a Calabi-Yau structure of degree d, one has an isomorphism

$$HH_i(\mathcal{C}) \cong HH^{d+i}(\mathcal{C})^\vee$$

between Hochschild homology and the linear dual of Hochschild cohomology, up to a shift of degree d.

(b) Let X be a smooth and proper Calabi-Yau variety of complex dimension d. Using Serre duality and the choice of a top-form on X, convince yourself that $D^bCoh(X)$ (or rather, an appropriate dg enbhancement) is Calabi-Yau of dimension d.

[24]Kevin Costello, "Topological conformal field theories and Calabi–Yau categories." Advances in Mathematics 210 (2007) 165–214.

Koszul duality exercises

Hiro lost steam; he will type these up as exercises in due time. For now, please have the beautiful hand-written set of exercises from Joey.

I-1

Koszul Duality

$(\mathcal{C}, \otimes)$ should be a category of (htpy) abelian groups.

Exs $(\mathcal{C}, \otimes) = (\mathrm{Vect}_k, \otimes_k)$ not htpcl but degl.

$(\mathcal{C}, \otimes) = (\mathrm{Ch}_R, \otimes_R)$

$(\mathcal{C}, \otimes) = (\mathrm{Spectra}, \wedge)$

All these settings make sense, but for the sake of ease these exercises, we focus on $\boxed{(\mathrm{Ch}_k, \otimes_k)}$ for k a field, char $k = 0$.

I. TREES

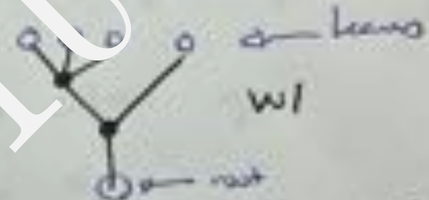
Recall $\mathbb{V}$ a symmetric seq in $\mathcal{C}$ is $\mathbb{V}: \mathrm{Fin}^{\cong} \to \mathcal{C}$.

Defn A poset $(P, \le)$ is a rooted tree iff

(1) $\forall x \in P$, the set $P_{x\le} := \{y \in P : x \le y\}$ is linearly ordered by $\le$.

(2) $\forall S \subseteq P$ s.t. $S \neq \emptyset$, the least upper bound of S exists.

Convention we draw trees



w/ internal vertices drawn as •, w/ leaves (least elements) and the root (greatest element) unlabeled.

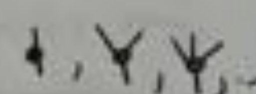
Examples

$\boxed{0}$ (0) Empty tree: ——

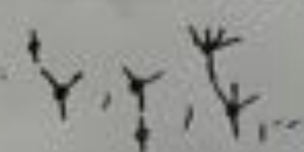
(1) Tree w/ no internal vertices:

Trees are "weight graded" by number of internal vertices

$\boxed{1}$ (2) Tree w/ one internal vertex:



$\boxed{2}$ (3) Tree w/ two " —— ":



Defn A planar tree is a triple $(P, \le, <_\ell)$ s.t.

(1) $(P, \le)$ is a rooted tree

(2) $<_\ell$ is a linear order of leaves$(P, \le)$

(3) $\forall x \in P$, leaves$(x) := \{ y \in \text{leaves}(P,\le) : y \le x \}$

is a convex subset of leaves$(P, \le)$ for $<_\ell$.

Ex (0)  empty tree

(1)  non-empty weight zero tree.

(2)  .

(3)  , ...

Exercise Come up w/ some non-examples.

<u>Defn</u> A ~~tree~~ (planar) tree is finite iff P is finite.

<u>Defn</u> Let $\Omega^{\simeq}$ be the category whose objects are finite planar rooted trees, & whose morphisms are isomorphisms of rooted trees (<u>not</u> necessarily planar).

<u>Ex</u> $T =$ 

$\Omega^{\simeq}(T,T) = \Sigma_n$.

<u>Ex</u> $T =$ 

$\Omega^{\simeq}(T,T) \hookrightarrow \Sigma_7$, w/ image $\Sigma_3 \times \Sigma_4 \xrightarrow[\text{perm}]{\text{block}} \Sigma_7$ given by block permutations.

<u>Ex</u> Convince yourself that for $T =$ 

$$\Omega^{\simeq}(T,T) \hookrightarrow \Sigma_{(n_1+n_2+\cdots+n_k)}$$

w/ image the same as

$$\Sigma_{n_1} \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_k} \xrightarrow[\text{perm}]{\text{block}} \Sigma_{(n_1+\cdots+n_k)}.$$

(Note for Hiro: maybe you want to say: <u>Ex</u> $\Omega^{\simeq}(T,T) \cong \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ & give $\Omega^{\simeq}(T,T) \hookrightarrow \Sigma_{\{\text{leaves}(T)\}}$ as the hint?)

Note Where does $\Omega^{\simeq}(T,T)$ appear in the defn of operad?

II. ~~[illegible]~~ [illegible] Free operads.

Defn · Category of operads $Op(\mathcal{C},\otimes)$

· Category of sym. seq. $\mathcal{C}^{\Sigma}$.

· Forgetful functor $Op(\mathcal{C},\otimes) \xrightarrow[U]{} \mathcal{C}^{\Sigma}$.

~~An operad $\mathcal{O}$ is free on a sym seq V if~~ Let $\mathcal{O}$ be an operad, V a sym. seq. We say a map of sym. seq. $V \xrightarrow[i]{} U\mathcal{O}$ exhibits $\mathcal{O}$ as a free operad on V iff: ~~[illegible]~~ P $\forall j \; \exists! \, \bar{j} : \mathcal{O} \to P$ s.t.

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & U\mathcal{O} \arrow[d, dashed, "U\bar{j}"] \\ & UP \end{tikzcd}$$

~~Defn We call a sym. seq./operad $\mathcal{O}$ 1-many iff for $n \leq 1$, $\mathcal{O}(n) = 0$.~~

Defn A sym seq V is 1-ary iff $\forall n \neq 1$ $V(n) = 0$. An operad is 1-ary iff its underlying sym. seq is 1-ary.

Ex Let $(\mathcal{C}^{\Sigma})_1$ denote the category of 1-ary sym seqs, and $\mathrm{Op}(\mathcal{C})_1$ denote 1-ary operads.

Prove $\mathrm{Op}(\mathcal{C})_1 \cong \mathrm{Monoid}(\mathcal{C}^{\Sigma}_1, \otimes)$

Hint $((\mathcal{C}^{\Sigma})_1, \circ) \cong (\mathcal{C}, \otimes)$.

Ex Prove that if V is a 1-ary sym. seq. the free operad on V exists & is given by

$$T(V) = V \oplus V \circ V \oplus V^{\otimes 3} \oplus \cdots$$

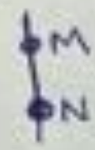
ie, the usual tensor algebra.

A general construction for free operad on sym seq V

The wrong idea: $V \oplus (V \circ V) \oplus (V \circ V \circ V) \oplus \cdots$

5

The Tensor Algebra formula works because
(products trivially commute)
$-\otimes-$ is linear in both variables; pictorially this is because it uses one copy of each term: $N \otimes M \leftrightarrow$



The circle product for operads is linear in the left variable, but not on the right: $N \circ M \leftrightarrow$

<u>Defn</u> Let V be a symmetric sequence; define $\mathrm{Pre}_V : \tilde{\Omega} \to \mathcal{C}$ by induction on weights of trees:

BASE CASES.

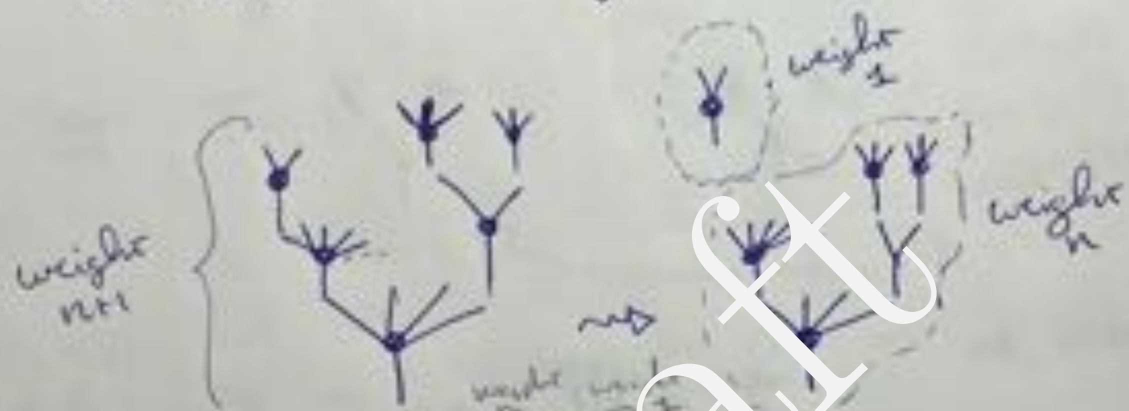
	WEIGHT 0		WEIGHT 1	
TREES T	$\emptyset$	$\vert$	$\bullet$	corolla, arity n
$\mathrm{Pre}_V(T)$	0	$1_{\mathcal{C}}$	0	$V(\mathrm{flags}\ T)$

6

Induction

Prev defined on weight n planar, rooted trees.

To extend to weight $n+1$, "pull off" a ~~corolla~~ corolla of leaves:



so that $T = T_1 \circ_i T_2$

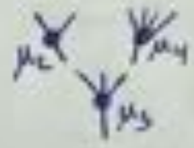
$$\mathrm{Prev}(T) = \mathrm{Prev}(T_1) \otimes (1_C^{\otimes} \otimes \mathrm{Prev}(T_2) \otimes 1)$$

i^{th} spot.

Exercise Use associativity of $\otimes$ to prove this construction is well-defined (or to make a defn that is!)

<u>Picture</u> For a tree T, think of $\mathrm{Pre}_V(T)$ [7]

as generated by T w/ labelled internal vertices, where labels come from V (w/ appropriate arity).

<u>ex</u> $T =$   $\mu_2 \in V(2)$, $\mu_3 \in V(3)$, $\mu_4 \in V(4)$.

<u>actual example</u> (1) Compute Pre_V for T as when

(i) $V = \mathrm{Com}$ (the underlying sym seq of commutative operad)

(ii) $V = \mathrm{As}$ (" ——— " associative operad).

<u>Note</u> (1) This definition uses PLANAR trees for Both symmetric & nonsymmetric sequences. For nonsymmetric sequences, we restrict to $\Omega^{\simeq}$ & consider only planar isomorphisms of trees.

18 10

(2) One can define a ~~[illegible]~~ symmetric sequence

$$\mathrm{PreFree}_V(n) = \coprod_{\substack{\text{iso classes } [T] \\ \text{of } \Omega^? \text{ whose} \\ \mathrm{leaves}(T)=n}} \mathrm{Prev}(T)$$

& define a composition structure

$$\mathrm{PreFree}_V \circ \mathrm{PreFree}_V \longrightarrow \mathrm{PreFree}_V$$

that is associative. Why does this composition fail to be an operad?

__Defn__ Let Free_V be the symmetric sequence defined by

$$\mathrm{Free}_V(n) = \coprod_{\substack{\text{iso classes } [T] \\ \text{of } \Omega^? \text{ whose} \\ \mathrm{leaves}(T)=n}} \mathrm{Prev}(T) \otimes_{\mathrm{Aut}(T)} \left(\cancel{\;\;}^{\otimes n} \right)$$

to see how $\mathrm{Aut}(T)$ acts, recall that $\mathrm{Aut}(T) \to \mathrm{Aut}(\mathrm{leaves}(T))$ from earlier.

9

10

Exercise Prove that the composition on

$$\mathrm{Free}_V \circ \mathrm{Free}_V \longrightarrow \mathrm{Free}_V$$

makes Free_V into an operad. Prove that Free_V has the universal property of the free left adjoint to

$$\mathcal{C}^{\Sigma} \xleftarrow{\text{forget}} \mathrm{Operad}(\mathcal{C}, \circ).$$

III. Cofree (conilpotent) cooperads $\mathcal{C}$

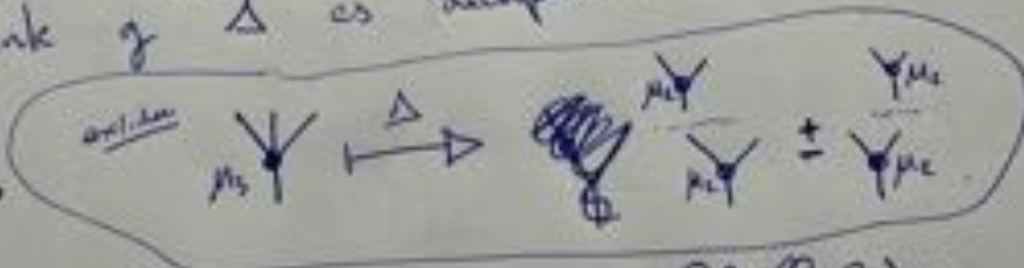
Defn A cofree conilpotent cooperad is a symm. seq V together w/ a map

$$V \xrightarrow{\Delta} V \circ V,$$

+ this is appropriately coassociative + symmetric.

Think of Δ as "decomposition"

this word makes me feel better at your expense. my apologies



Prop For a sym. seq V, Free_V is also the conilpotent cofree cooperad:

$$\mathrm{Free}_V \longrightarrow \mathrm{Free}_V \circ \mathrm{Free}_V$$

defined by "pulling off one corolla at a time" and summing:

coOperad

forget $\downarrow \dashv \vdots$ cofree conil cooperad

$\mathcal{C}\Sigma$

is RIGHT adjoint to forget.

Heuristic.

* let's say finite arity-wise

Ex's The linear dual of any finite operad* is a cooperad.

(0) Compute the dual cooperad to Com.

(1) Compute the dual cooperad to As.

Lie? CoLie?

LECTURE VI

∞-categories

"It is time."
– Rafiki

My goal today is to finally peek under the hood of this thing called ∞-categories. I want to convey that fancy-schmancy homotopical arguments are actually enabled through incredibly concrete combinatorics. The big victory is that we have one language for organizing intricate homotopical structures, and that this language has incredible formal propertes.

Indeed, the success of ∞-categories in the last two decades isn't really a victory of higher category theory – I think people already thought about higher-categorical structures. The real victory is for combinatorics. The reason ∞-categories are useful is that there is a concrete, unambiguous, combinatorial way to prove things.[1]

Remark VI.0.0.1 (References). One of the challenges as someone starting to learn this stuff is that – even though everything is written down – it is hard to know which of Jacob Lurie's lemmas/propositions accomplish what we want to accomplish. So I have written references where appropriate.

So if you see anything references below like "Proposition 1.1.2.2" you should know that this refers to Proposition 1.1.2.2 in Lurie's Higher Topos Theory. The PDF of Higher Topos Theory is freely available on Lurie's website[2].

I may also reference Lurie's second book, Higher Algebra. This is also freely available on Lurie's website[3].

VI.1. Categories and their nerve

Let $\mathcal{C}$ be a category (in the usual sense). For sake of having a concrete example, you can imagine that $\mathcal{C}$ is the category of groups.

I want to explain how, from $\mathcal{C}$, I can extract a completely combinatorial object. This comes down to thinking of a category as a bunch of marshmallows, toothpicks, rubber triangles, and higher-dimensional versions of

[1] This is not to say that other attempts are higher categories were ill-defined. They were perfectly well-defined. They were simply harder to work with for certain purposes.

[2] `https://www.math.ias.edu/~lurie/papers/HTT.pdf`

[3] `https://www.math.ias.edu/~lurie/papers/HA.pdf`

these. Another way to think about the combinatorics is: How can you draw a cartoon of a category?

Well, let me draw a vertex (a marshmallow) for every object of $\mathcal{C}$. So I have a set[4] of vertcies in bijection with the set of finite groups. Because vertices are 0-dimensional objects, I will write this set of vertices as

$$\mathcal{C}_0.$$

Then, every time I have a morphism (e.g., a group homomorphism) from X to Y, I am going to draw a toothpick – an oriented toothpick, mind you – from the vertex X to the vertex Y. Note that a given pair of vertices may have many, many arrows between them.[5] A combinatorial way to encode all this is that we have a set of edges

$$\mathcal{C}_1$$

which is in bijection with the set of all morphisms in $\mathcal{C}$, and I have a prescription of where to send the head and tail of the edges:

$$d_0 : \mathcal{C}_1 \to \mathcal{C}_0, \qquad d_1 : \mathcal{C}_1 \to \mathcal{C}_0.$$

d_0 picks out the head, and d_1 picks out the tail. The i in d_i will make sense in a moment. Note that if an object has endomorphisms (all objects do – there are identity morphisms) then we have "loop" edges accordingly. This won't be a big deal at all for us.

So far, we have extracted the structure of a directed graph from a category. Let me start using the fact that there's a composition in my category. That is, given morphisms $f_{12} : X_1 \to X_2$ and $f_{01} : X_0 \to X_1$, I know exactly what I mean by the composition $f_{12} \circ f_{01} : X_0 \to X_2$.

So, let me attach a triangle between three edges $f_{ij} : X_i \to X_j$ precisely when I know the following:

$$f_{12} \circ f_{01} = f_{02}.$$

In other words, in my combinatorial gadget, I will draw insert a triangle for every commutative triangle[6] in my category $\mathcal{C}$. I will call the set of such triangles

$$\mathcal{C}_2.$$

[4]There is of course the real, important observation that the "set of all sets" (and in particular, the set of all abelian groups) is not a set. There are standard fixes to this observation, the most common being the use of Grothendieck universes. If you are comfortable with Grothendieck universes (i.e., assuming that inaccessible cardinals exist), you can of course take categories of whatever size you like; then the common language is that the "set of all (small) sets" is not a small set, but is a "large" set.

[5]Because any two groups always admit a homomorphism between them in either direction, this particular graph happens to be connected. But generally, these graphs may be disconnected; even when the graph is connected, two objects may not have any morphisms between them.

[6]Commutative triangles do not get as much love as commutative squares. Today is their day.

Note that $\mathcal{C}_2$ then has three maps

$$d_0, d_1, d_2 : \mathcal{C}_2 \to \mathcal{C}_1$$

telling us how to glue a given triangle to three edges. Let me explain the indexing. Because our triangles should be thought of as oriented, we have an ordering on its set of vertices, so we can talk about the 0th, 1st, and 2nd vertices of a triangle. Then d_i says "the edge *opposite* the ith vertex should be glued to this edge."

Example VI.1.0.1. For example, given a commutative triangle

$$T = \begin{tikzcd} X_0 \arrow[r, "f_{01}"] \arrow[dr, "f_{02}"'] & X_1 \arrow[d, "f_{12}"] \\ & X_2 \end{tikzcd} \quad \in \mathcal{C}_2$$

then

$$d_0(T) = f_{12}, \qquad d_1(T) = f_{02}, \qquad d_2(T) = f_{01}.$$

You can do the same thing for any $k \geq 3$. We declare

$$\mathcal{C}_k$$

to be the set of commutative k-simplices in $\mathcal{C}$. Let me run through, very explicitly, what I mean by a commutative k-simplex. I mean a collection of objects $X_0, \ldots, X_k$, morphisms $f_{i,j} : X_i \to X_j$ for $i < j$, such that

$$f_{i,k} = f_{j,k} \circ f_{i,j}$$

for every triplet $i < j < k$. Let me assure you this isn't very much data – it's uniquely specified by the consecutive edges $f_{01}, f_{12}, f_{23}, \ldots, f_{(k-1)k}$. Again, $\mathcal{C}_k$ has $k+1$ "boundary" functions:

$$d_0, d_1, \ldots, d_k : \mathcal{C}_k \to \mathcal{C}_{k-1}$$

telling us which $(k-1)$ simplex we should glue the face opposite the ith vertex.

So far, we witness the combinatorics of maps that look as follows:

$$\cdots \begin{tikzcd} {} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathcal{C}_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{C}_2 \arrow[r, shift left=1] \arrow[r, shift right=1] & \mathcal{C}_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & \mathcal{C}_0 \end{tikzcd} \tag{VI.1.1}$$

But there's *more*!

Categories have things called identity morphisms for objects. That is, for every object X, there is a distinguished morphism id_X so that $\mathrm{id}_X \circ f = f$ (or $g \circ \mathrm{id}_X = g$) for any morphism f (or g) with codomain (or domain) given by X. So, there's a map

$$\mathcal{C}_1 \leftarrow \mathcal{C}_0,$$

sending a vertex X to the directed edge id_X. And indeed, given any commutative k-simplex, we can make a $k+1$-simplex by inserting an identity edge at any vertex. Let me illustrate this in two examples:

$$\left(X_0 \xrightarrow{f} X_1\right) \mapsto \left(\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[dr] & X_1 \arrow[d, "\mathrm{id}_{X_1}"] \\ & X_1 \end{tikzcd}\right) \quad \text{or} \quad \left(\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}_{X_0}"] \arrow[dr] & X_0 \arrow[d, "f"] \\ & X_1 \end{tikzcd}\right)$$

We will call these two triangles $s_1 f$ and $s_0 f$, respectively, where the s_i means we insert an identity morphism from the ith vertex to a new vertex, to be interpreted as the $(i+1)$st vertex. What we then witness are k maps

$$\mathcal{C}_k \leftarrow \mathcal{C}_{k-1} : s_i, \qquad i = 0, \ldots, k-1.$$

Drawing all of these for all k at once, we have maps that look as follows:

$$\text{(VI.1.2)} \qquad \begin{tikzcd} \ldots & \mathcal{C}_3 \arrow[l, bend right] \arrow[l, shift right] \arrow[l, shift left] \arrow[l, bend left] & \mathcal{C}_2 \arrow[l, shift left] \arrow[l] \arrow[l, shift right] & \mathcal{C}_1 \arrow[l, bend right] \arrow[l, bend left] & \mathcal{C}_0 \arrow[l] \end{tikzcd}$$

Definition VI.1.0.2 (Face and degeneracy maps). The maps depicted in the original diagram (VI.1.1) are called *face maps* (because they pick out the faces of a commutative k-simplex) and the maps in (VI.1.2) are called *degeneracy* maps, because the images of the maps feel "degenerate" by virtue of being results of identity morphism insertions.[7]

Remark VI.1.0.3. If you are to remember any diagram, it is (VI.1.1). All the meat of anything we study today will be in the face maps.

What we see is that there is some potentially structured combinatorial data arising from any category. It's time to organize this data.

VI.2. Simplicial sets

The beautiful fact is that the above combinatorics can be encoded with great efficiency.

Notation VI.2.0.1. For a given integer $n \geq 0$, we let

$$[n] = \{0 < 1 < \ldots < n\}$$

denote the linearly ordered set with $n+1$ elements.

Remark VI.2.0.2. Recall that a map of posets is a map $f : P \to Q$ for which $p \leq p' \implies f(p) \leq f(p')$. A map f of posets is an isomorphism if f is bijection whose inverse is also a map of posets. Then any finite, non-empty, linearly ordered poset is isomorphic to a $[n]$ for some unique n.

[7]Warnings: If you are new to this game, beware that the d_i are the face maps; *not* degeneracy maps, despite the fact that "d" is the letter that begins degeneracy. You should imagine that d_i is being used because d is often used for boundaries in algebra. As we will see soon, the s_i are dual to *surjections*.

Definition VI.2.0.3 (The simplex category). We let

$$\Delta$$

denote the category whose objects are finite, non-empty, linearly ordered sets. Morphisms are map of posets.

Remark VI.2.0.4. Perhaps this follows Kate's observation that too many things are labeled by Δ. But I think it is fair to say that the use of Δ to denote simplices, and the above category of simplices, is a very important and common notation – and in many ways is more fundamental than most mathematical instances of the notation Δ, perhaps with the exception of "change" in calculus.

Remark VI.2.0.5. By Remark VI.2.0.2, we may pretend that Δ is a category (up to equivalence) whose objects are literally in bijection with the non-negative integers: There is an object $[n]$ for every $n \geq 0$.

Example VI.2.0.6. Fix k. Then there are $k+1$ injections

$$\delta_i : [k] \to [k+1]$$

given by "skipping the ith element" of $k+1$. Drawing only the injections between consecutive k, we thus find a diagram whose shape is dual to (VI.1.1).

$$\begin{tikzcd} \cdots & {[3]} \arrow[l, shift left=4] \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[l, shift right=4] & {[2]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & {[1]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & {[0]} \arrow[l, shift left=1] \arrow[l, shift right=1] \end{tikzcd}$$

Likewise, there are k surjections

$$\sigma_i : [k] \to [k-1]$$

given by "send the ith and $i+1$st elements to the same image." You can draw the corresponding diagram.

Remark VI.2.0.7. We can also understand every morphism in Δ by understanding the injections and surjections, as every map will factor uniquely as a surjection followed by an injection.

What we find is that the combinatorial data we observe in $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ looks exactly *dual* (i.e., arrows reversed) to the combinatorial data of Δ. This inspires us to define the following:

Definition VI.2.0.8. A *simplicial set* is a functor

$$\Delta^{\mathrm{op}} \to \mathsf{Sets}.$$

A map of simplicial sets is a natural transformation. We let

$$\text{sSets}$$

denote the category of simplicial sets.

Remark VI.2.0.9. Because all functions are generated by injections and surjections, and because it turns out all surjections in Δ are compositions of σ_i maps, and all injections are compositions of δ_i maps, a simplicial set can be determined by exactly the data of

- A set X_k for every $k \geq 0$,
- Functions $d_0, \ldots, d_k : X_k \to X_{k-1}$ for every $k \geq 1$, and
- Functions $s_0, \ldots, s_k : X_k \to X_{k+1}$ for every $k \geq 0$.

These functions do need to satisfy some relations called the *simplicial relations*. There are five of them, so yes, this is dirty work, but it's worth doing:

(1) $d_i d_j = d_{j-1} d_i$ for $i < j$.
(2) $s_i s_j = s_{j-1} d_i$ for $i < j$.
(3) $d_i s_j = \mathrm{id}$ if $i = j$ or $i = j + 1$.
(4) $d_i s_j = s_j d_{i-1}$ if $i > j + 1$.
(5) $s_i s_j = s_{j+1} s_i$ if $i \leq j$.

Example VI.2.0.10 (The nerve of a category). Fix a category $\mathcal{C}$. The data of the face and degeneracy maps from earlier define a simplicial set sending

$$[k] \mapsto \mathcal{C}_k, \qquad \delta_i \mapsto d_i, \qquad \sigma_i \mapsto s_i.$$

This simplicial set is called the *nerve* of the category $\mathcal{C}$.

Notation VI.2.0.11 ($N(\mathcal{C})$). It is an eternal struggle in this business whether to notate the nerve using $\mathcal{C}_0, \mathcal{C}_1, \ldots$ or to use a symbol like

$$N(\mathcal{C})$$

to denote the nerve, to remind us that $\mathcal{C}$ is a category and $N(\mathcal{C})$ is a simplicial set. I may be sloppy with this from time to time.

Remark VI.2.0.12. Moreover, to give a functor between two categories $\mathcal{C}$ and $\mathcal{D}$ is the same thing as giving a map of simplicial sets between their nerves. More precisely, the nerve functor is a fully faithful embedding of the category of categories into the category of simplicial sets – Exercise VI.15.

You should think of "simplicial sets" as a purely combinatorial object – the data required to produce a simplicial set can be checked using precise equalities, and these equalities are manageable.

Thus, this remark should be thought of as saying: "The theory of categories is completely encoded by combinatorics." This will be more true in a moment – the notion of natural transformation of functors, for example, has yet to be discussed.

Remark VI.2.0.13. A natural transformation of simplicial sets is a concrete piece of data one can often write down. Given two simplicial sets X and Y, a map from X to Y is just the data of a function $f_k : X_k \to Y_k$ for every $k \geq 0$. It turns out every morphism in Δ is a composition of the σ_i and δ_i, so one only needs to check that the f_k commute with the s_i and the d_i (that is, that f respects fact and degeneracy maps).

VI.3. Simplicial sets and spaces

So, it's pretty neat that every category can be encoded, quite nicely and faithfully, as a simplicial set (a combinatorial object). But here's where the theory is even better: A simplicial set can also encode a topological space.

Example VI.3.0.1 (The singular chains of a space). Let X be a topological space. We define a simplicial set

$$\mathsf{Sing}(X)$$

as follows. The set of k-simplices

$$\mathsf{Sing}(X)_k := \hom_{\mathsf{Spaces}}(\Delta^k, X)$$

is given by the *set* of continuous functions from the standard k-simplex to X.

Here's what we can observe: There are standard inclusions of the faces of a k-simplex into Δ^k, which we might as well call ∂_i. Likewise, there are linear projection maps from a k-simplex to the $(k-1)$-simplex, which we may as well call σ_i. Thus, pulling back along the ∂_i and σ_i gives rise to the d_i and s_i of $\mathsf{Sing}(X)$.

Let's digest this simplicial set a bit. The set $\mathsf{Sing}(X)_0$ is in bijection with X itself. Then, the set $\mathsf{Sing}(X)_1$ is in bijection with the set of continuous paths in X. Given $\gamma \in \mathsf{Sing}(X)_1$, the boundary $\partial_1\gamma$ is the starting point of the path, and $\partial_0\gamma$ is the ending point.

Example VI.3.0.2. Now, suppose that we have two paths γ_{12}, γ_{01} where γ_{ij} is a path from x_i to x_j. Then the data of a 2-simplex $H : \Delta^2 \to X$ with

$$\partial_0 H = \gamma_{12}, \partial_2 H = \gamma_{01}$$

would precisely be the following data: A new path called $\partial_1 H$, and a *homotopy* between the endpoint concatenation $\gamma_{12} \circ \gamma_{01}$ and $\partial_1 H$.

I am using the notation $\circ$ here, not $\sharp$. One is to connote composition, but the other is to really emphasize that $\gamma_{12} \circ \gamma_{01}$ is *not* parametrized by a single 1-simplex; it is rather parametrized by a "horn" obtained by gluing two 1-simplices along a common endpoint. So there is no need to choose any reparametrizations of paths in this interpretation.

You should thus interpret H as a "homotopy comuting triangle" in some category. $\mathsf{Sing}(X)$ is like a category whose objects are given by elements of X, whose morphisms are given by continuous paths (from 1-simplices) in X, but where we do *not* define what composition is, strictly speaking. (Indeed, we do not choose to interpret $\gamma_{12} \circ \gamma_{01}$ as a data of a new 1-simplex; it just is what it is.) We simply say that there exist certain 2-simplices, and interpret them as datum exhibiting that there is some notion of homotopy between a potential composition of the γ_{ij}.

VI.4. Kan complexes (the "spaces") of simplicial sets

It turns out that one can create a homotopy theory of spaces using only simplicial sets. This was one of the original motivations for people like Kan and Quillen to study simplicial sets. But to initiate such a study, it is convenient to pick out the simplicial sets that behave like spaces. (As an example, not every category behaves like a space.) Such simplicial sets are called *Kan complexes*, and to define what a Kan complex is, I need to define horns.

Notation VI.4.0.1 (The n-simplex as a simplicial set). We call the simplicial set

$$\Delta^n = \hom_\Delta(-, [k])$$

the simplicial n-simplex. (This double-books our notation for the topological n-simplex; we will live with this.) See Exercise VI.17 for more exposition on this.

Remark VI.4.0.2. By the Yoneda lemma, a map of simplicial sets $\Delta^n \to X$ is the exact same thing as picking out an element of X_n; i.e., as an n-simplex of X. See Exercise VI.17.

Remark VI.4.0.3. The above formula can seem a bit abstract if you don't like the Yoneda embedding, so I would encourage you to simply think of Δ^n as the nerve of the poset $[n]$, which probably feels more finite. (This nerve is isomorphic to Δ^n.)

In fact, realizing that Δ^n is not only the functor represented by $[n]$, but is also the nerve of $[n]$, is incredibly powerful. The Yoneda embedding allows us to conclude that simplicial set maps from Δ^n are precisely the same thing as picking out an n-simplex of the target. On the other hand, thinking of $[n]$ as some free category generated by n consecutive morphisms, we have hope that maps out of Δ^n encode simple categorical building blocks.

Notation VI.4.0.4 (Horns). Fix $n \geq 0$ and $0 \leq k \leq n$. We let

$$\Lambda^n_k \subset \Delta^n$$

denote the simplicial set obtained from Δ^n by deleting the (interior of) the n-simplex, and deleting the face of Δ^n opposite the kth vertex.

Remark VI.4.0.5 (Expositing Λ^n_k). The above definition of Λ^n_k is a bit informal. Here is a more concrete definition: We define

$$\Lambda^n_k([a])$$

to be the set of poset maps $[a] \to [n]$ that do *not* surject onto the subset $[n] \setminus \{k\}$.

Probably the most healthy characterization of Λ^n_k is as the simplicial set glued out of exactly n simplices of dimension $n-1$; they are glued together exactly the way that the faces of Δ^n are glued together, and without including the kth face.

Example VI.4.0.6. Below are the pictures of the three possible horns for $n = 2$:

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd} .$$

These are the horns $\Lambda^2_1, \Lambda^2_0, \Lambda^2_2$ respectively.

Remark VI.4.0.7 (Horns can't always be filled in a category). Note that if the above diagrams depicted morphisms in a category, only Λ^2_1 has any hope of being completed to a 2-simplex; the others could only be completed if there were a nice factorizatoin property, or perhaps inverses, to the given edges/morphisms.

Remark VI.4.0.8 (Horns can always be filled for a space). On the other hand, imagine being given a continuous map from a horn-shaped space to a space X. The horn-shaped space is a strong retract of a simplex; thus, we can "extend" any continuous map from a horn to a continuous map from the simplex. This inspires the following definition.

Definition VI.4.0.9. Let X be a simplicial set. We say that X is a *Kan complex* if for every $n \geq 0$ and every $0 \leq k \leq n$, every map from the horn

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

extends to a map from the n-simplex.

Example VI.4.0.10. Let W be a topological space. Then $\mathsf{Sing}(W)$ is a Kan complex.

Remark VI.4.0.11. Given a map from a horn, the filler from the simplex is rarely unique. Make sure you understand this point in the example of Sing.

Example VI.4.0.12. A simplicial group is a functor $\Delta^{\mathrm{op}} \to \mathsf{Ab}$ to the category of groups. By forgetting the abelian groups to simply be sets, one obtains a simplicial set. It turns out any simplicial group is a Kan complex – Exercise VI.19.

Remark VI.4.0.13. If Kan complexes are like spaces, there should certainly be a notion of homotopy groups of a Kan complex, again defined using the language of simplicial sets. This is given in Exercise VI.18.

Remark VI.4.0.14. For more on the homotopy theory of simplicial sets (modeling topological spaces), I recommend the book of Goerss-Jardine[8].

[8]Paul G. Georss and John F. Jardine, "Simplicial Homotopy Theory." Reprint of the 1999 original. Basel: Birkhäuser (2010)

VI.5. Categories inside sSets

In fact, we can also characterize (simplicial sets arising as nerves of) *categories* inside all simplicial sets.

THEOREM VI.5.0.1. [Proposition 1.1.2.2] Let X be a simplicial set. Then X is isomorphic to the nerve of a category if and only if, for every map from an *inner horn*, there exists a *unique* filler to a map from a simplex. That is, for every $n \geq 2$ and $0 < k < n$, and every map $\Lambda^n_k \to X$, the filler below uniquely exists:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "!\exists"'] & \end{tikzcd}$$

REMARK VI.5.0.2. For the case $n = 2, k = 1$, you should think of the map from Λ^2_1 as giving rise to two morphisms call f_{12}, f_{01}. That there exists a unique extension to the 2-simplex means that there is a unique edge one ought to call the *composition* of the two given morphisms, and that there is a unique 2-simplex realizing this (as a commutative diagram). In particular, if there is any commutative diagram involving f_{01}, f_{12}, and a third edge f_{02}, then one can conclude that $f_{02} = f_{12} \circ f_{01}$.

VI.6. ∞-categories

I have tried to convince you that this purely combinatorial idea of a simplicial set can capture topological spaces (via Sing) and categories (via the nerve).

Staring at the definition of Kan complex, and the characterization of categories, one is led to contemplate whether a existence (but not uniqueness) of horn-fillings are a natural thing to look at.

For example, suppose that you could fill every horn Λ^2_1 to Δ^2, but not necessarily uniquely. Taking the hint from 2-simplices in $\mathsf{Sing}(W)$, one could interpret each horn-filling 2-simplex as exhibiting a third edge (i.e., a morphism) f_{02}, and a statement that f_{02} is homotopic to "some composition of f_{12} with f_{01}." But a change in perspective arises: We never do need to define composition on the nose; the horn-filling just tells us we can produce third edges out of 2, and in perhaps many ways.

That the following condition is important was first identified by Andre Joyal – he used the term "quasi-category" to describe these simplicial sets. Lurie is responsible for the term "∞-category."

DEFINITION VI.6.0.1 (∞-category). Let $\mathcal{C}$ be a simplicial set. We say that $\mathcal{C}$ is an *∞-category* if, for every $n \geq 2$ and for every $0 < k < n$, any map from Λ^n_k into $\mathcal{C}$ extends to the n-simplex:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

We see that this definition broadens Kan complexes, in the sense that horn-fillers need only exist for certain horns. This reflects the fact that not every morphism in a category is invertible. (See Remark VI.4.0.7.)

The definition of ∞-category also broadens categories; as already discussed, the notion that these horns may be filled, but not uniquely, opens the door to interpreting simplices as homotopy coherent diagrams, and to the fact that a given collection of morphisms may have many different ways to produce new morphisms.

Definition VI.6.0.2. Let $\mathcal{C}$ be an ∞-category. We call an element of $\mathcal{C}_0$ (i.e., a vertex) an *object* of $\mathcal{C}$. We call an element of $\mathcal{C}_1$ (i.e., an edge) a *morphism*.

Given a morphism $f \in \mathcal{C}_1$, we say that $x_1 = d_0 f$ is the codomain of f, and $x_0 = d_1 f$ is the domain of f. Alternatively, we say that f is a morphism from x_0 to x_1.

Remark VI.6.0.3. An ∞-category was originally called a *quasi-category* by Joyal. Boardman and Vogt called it a *weak Kan complex*. The term ∞-category originates in Lurie's writing.

Be warned that the term ∞-category, in conversation, can refer to the idea of $(\infty, 1)$-category in general[9]. For us, in these lectures, ∞-category is the notion given in Definition VI.6.0.2.

Remark VI.6.0.4. Out of any ∞-category $\mathcal{C}$, there are two natural Kan complexes one can define.

One is the "largest Kan complex sitting inside $\mathcal{C}$." Informally, this Kan complex is obtained by throwing out all non-homotopy-invertile morphisms from $\mathcal{C}$. Lurie often denotes this Kan complex by

$$\mathcal{C}^{\simeq}.$$

The other is the "smallest Kan complex containing $\mathcal{C}$." An informal combinatorial description is as follows: This Kan complex is obtained inductively by attaching more and more simplices so that all horns have a filler. Categorically, an informal description is that this Kan complex is obtained by localizing all morphisms of $\mathcal{C}$ (i.e., rendering all morphisms of $\mathcal{C}$ invertible up to homotopy). I often denote this ∞-category by

$$|\mathcal{C}|$$

[9]That is, a "category" containing notions of higher k-morphisms for all $k \geq 1$, and for which every morphism of level $k \geq 2$ is invertible up to higher morphism.

but I must admit that this is not great notation. It is meant to connote the geometric realization of $\mathcal{C}$, which is an actual topological space, but whose Sing does model the Kan completion of $\mathcal{C}$. See Exercise VI.21.

VI.7. Functors

Definition VI.7.0.1. A map of ∞-categories is a map of simplicial sets. We will also call such a thing a *functor* between ∞-categories.

Remark VI.7.0.2. Let Δ^n be the nerve of $[k]$. You should think of this as a very rigid, classical category.

Fix an ∞-category $\mathcal{C}$. Then a functor $\Delta^n \to \mathcal{C}$ is now, all of a sudden, an incredibly homotopical object. The edges of Δ^n specify morphisms in $\mathcal{C}$, but all its simplices up to dimension n now encode homotopies and higher homotopies between "compositions" of these morphisms.

One way to think about this is that producing a functor might be difficult, because a functor will contain so much data. In reality, it is true that it can take a lot of work to make certain functors; but let's keep in mind that every functor in the usual sense gives rise to maps of simplicial sets – Exercise VI.15. So in fact, this new notion of functor allows us to do strictly more than we could before.

Even better, the notion of "map of simplicial sets" gives a very concrete definition for what it means to produce a homotopy coherent functor; you could imagine that such a notion spent decades enjoying intuition but lacking definition.

Remark VI.7.0.3. Another point is that – while individual functors may be hard to write down – the collection of all functors is incredibly formal to write down; and indeed, the ∞-category of functors is easy to write down. See Exercise VI.23.

Indeed, in that same exercise you will be asked to play with natural transformations as well; they are defined quite easily.

Example VI.7.0.4. Let $\mathcal{D} = \mathsf{Sing}(W)$ be (the singular complex of) a topological space. We can now speak of functors $\mathsf{Sing}(W) \to \mathcal{C}$, and of limits/colimits of such functors – this is quite exciting! For example, if $W = BG$ is the classifying space of a group G, a functor $\mathsf{Sing}(W) \to \mathcal{C}$ exactly encodes the data of an object of $\mathcal{C}$ with a homotopy coherent G-action (even when G is a discrete group!).

Indeed, when G is a discrete gropu, you can literally write down a simplicial set called BG – the set of k-simplices is given by G^k (so there is a unique 0-simplex), and the face maps are given by forgetting or multiplying factors. This simplicial set is a Kan complex. It models the usual classifying space.

VI.8. Examples

There are so many that they are quite hard to write down in one talk. But a lot of examples of categories that ought to be "higher categories" are known to naturally fit into the framework of ∞-categories.

REMARK VI.8.0.1. While this lecture uses ∞-categories, depending on the context, sometimes other models of higher categories are more advantageous. The model of *complete Segal spaces* are also incredibly useful.

The technology of model categories is incredibly useful for computations, but a bit clunky, because it can only model presentable ∞-categories; in particular, one needs all limits and colimits to exist. For example, the ∞-category Δ^k does not exist as a model category.

The model of *categories enriched in topological spaces* is the most intuitive, but turns out to be incredibly difficult to work with. Trying to enrich such a collection over itself (what is the Top-enriched category of functors between two topologically enriched categories?) is do-able but suffers from the reality that enrichments often give rise to incorrect mapping spaces.

Again, a useful analogy might be that a given module or chain complex admits many different resolutions. Depending on what you want to do, different resolutions are better than others. For example, Koszul resolutions are fantastic for computation, but very dependent on context and hence not great for relating different algebras. On the other hand, the bar resolution can be difficult to compute with, but gives functorial properties very nicely. This is also similar to the difference between cellular chains and singular chains. (I hope you have never computed the homology of a space using only the definition of singular chains; we rather use their formal properties.)

EXAMPLE VI.8.0.2. We have already seen that every category is an example of an ∞-category (by taking the nerve).

EXAMPLE VI.8.0.3 (Spaces and ∞-groupoids). We have also seen that every space is an example of an ∞-category (by taking Sing). This is an example where you should think of every morphism as invertible – i.e., as an ∞-*groupoid*. This is because any path has an inverse up to homotopy; this is a many-basepoint version of the fact that π_1 is a group.

DEFINITION VI.8.0.4. An ∞-*groupoid* is a Kan complex. (Informally, an ∞-groupoid is an ∞-category where every morphism is invertible up to homotopy.)

EXAMPLE VI.8.0.5 (Topologically enriched categories). Let C be a category enriched in topological spaces, so that for every pair of objects, $\hom_C(x, y)$ is given the structure of a space.

Then there is an associated ∞-category called the "homotopy coherent nerve", which I will also write as $N(C)$. This nerve is characterized by properties I will try to write out in Exercise VI.25; let me give some examples of simplices.

A 0-simplex of $N(C)$ is an object of C.
A 1-simplex of $N(C)$ from x to y is a point of $\hom(x, y)$.
A 2-simplex of $N(C)$ is the choice of

- a triplet of objects x_0, x_1, x_2,
- choices of points $f_{i,j} \in \hom(x_i, x_j)$ for $i < j$,
- and a choice of path in $\hom(x_0, x_2)$

$$f_{12} \circ f_{01} \sim f_{02}.$$

  Note here we are using the topology in $\hom(x_0, x_2)$, and the definition of composition given from C.

Higher simplices are where the meat is at; these will be exposited in the exercise.

Example VI.8.0.6 (The ∞-category of spaces). Let Top be the category of spaces that are, abstractly, homotopy equivalent to CW complexes. Then Top is enriched over the category of topological spaces. By applying the homotopy coherent nerve construction, we obtain an ∞-category Spaces. We call this the ∞-category of spaces. This might feel unnatural to you; see Section VI.10

Example VI.8.0.7 (The ∞-category of ∞-categories). Let C be the category of ∞-categories – objects are simplicial sets that happen to be ∞-categories, and morphisms are maps of simplicial sets. Then C can be enriched over simplicial sets – given two objects X and Y, the simplicial set

$$\{\hom_{\mathsf{sSet}}(X \times \Delta^k, Y)\}_{k \geq 0}$$

turns out to be a Kan complex. Then one can take the coherent nerve of this Kan-complex-enriched category. We call this nerve the *∞-category of ∞-categories*, and denote it

$$\mathcal{C}\mathrm{at}_\infty .$$

Yes, there are size issues – meaning yes, they can be overcome in the standard ways. I think the most popular method is by utilizing the theory of Grothendieck universes so we know what we mean by small, big, large, huge, et cetera, sets.[10]

Example VI.8.0.8 (dg- and A_∞-categories). Let $\mathcal{A}$ be a dg-category. One can define an ∞-category, called the *dg-nerve* of $\mathcal{A}$

$$N_{dg}(\mathcal{A})$$

as follows.[11]

The 0-simplices of $N_{dg}(\mathcal{A})$ are the objects of $\mathcal{A}$.

[10]That Grothendieck universes – i.e., a collection of "sets" closed under natural operations – exist is equivalent to the axiom of existence of inaccessible cardinals. This is a common assumption for many logicians, though I do not know how popular it is in the large.

[11]This construction is due to Lurie and can be found in his Higher Algebra book, available freely on this website.

A 1-simplex of $N_{dg}(\mathcal{A})$ is the data of a triplet

$$(x, y, f)$$

where x, y are objects and $f \in \hom_0(x, y)$ is a degree 0 element of the hom chain complexes for which $df = 0$.

A 2-simplex of $N_{dg}(\mathcal{A})$ is the data of a tuple

$$(x_0, x_1, x_2, f_{i,j} : x_i \to x_j (i \leq j), H)$$

where the $f_{i,j} : x_i \to x_j$ are 1-simplices as before, for $i < j$, and $H \in \hom_1(x_0, x_2)$ of degree 1 for which

$$dH = f_{02} - f_{12} \circ f_{01}.$$

The higher-dimensional simplices are spelled out in Exercise VI.26. The A_∞-analogue is due to Faonte and to Tanaka independently.

VI.9. (Not covered in spoken lecture) The downside: degeneracy maps

There is, objectively, one very annoying part of the definition of ∞-categories. Because an ∞-category is a simplicial set, one must specifies degeneracy maps. In particular, for any object $X \in \mathcal{C}_0$, one must provide what one means to be "the" identity morphism of X.

This is dissatisfying for the following reason: Units do not need to be unique; they are only unique up to homotopy. Moreover, whether something is unital is a *property*, not extra data to specify.

On the other hand, it is known from the classical theory of simplicial sets that degeneracy maps are incredibly useful for relating the combinatorics of simplicial sets to the homotopy theory of spaces. In a world without degeneracies, the homotopy type of the direct product $X \times Y$ will not recover the direct product of the homotopy types modeled by X and Y.

There are solutions to this issue. Steimle[12] showed that if a "simplicial set with only face maps" (known as a semisimplicial set) looks like an ∞-category, then indeed you can always equip it with degeneracy maps. And Tanaka[13] showed a similar result at the level of functors – if a map that only respects face maps looks like it respects units, you can always define a new map that does respect units, and respects the original images up to homotopy.

I think it is fair to say that no model is perfect.

[12] Wolfgang Steimle, "Degeneracies in quasi-categories." J. Homotopy Relat. Struct. (2018) 13:703–714. `https://doi.org/10.1007/s40062-018-0199-1`.

[13] Hiro Lee Tanaka, "Functors (between ∞-categories) that aren't strictly unital." Journal of Homotopy and Related Structures (2018) 13: 273–286. doi: 10.1007/s40062-017-0182-2. arXiv:1606.05669.

VI.10. (Not covered in spoken lecture) Making important ∞-categories (spaces, chain complexes)

In Example VI.8.0.6 we constructed an ∞-category of spaces, but there was something choice-dependent: Why did we choose (spaces homotopy equivalent to) CW complexes as our objects? What if we want to consider other objects?

There are indeed *two* common ways to make ∞-categories of very well-studied classical categories.[14] In such well-studied settings, one does often single out a nice class of objects. So one approach is to make an ∞-category by applying the coherent nerve to a topologically enriched category of nice objects.

Here is the second approach: One takes the category of all objects, nice or not (e.g., all topological spaces). Then one *localizes* this category with respect to all the equivalences we care about (e.g., weak homotopy equivalences). It turns out that this localization is almost always equivalent to the ∞-category constructed by the "nice objects" approach.

Remark VI.10.0.1. A subtle point here is that there is a natural notion of localization whose input may be a category in the usual sense (i.e., have no higher homotopy and all horn-fillers are unique) but whose output is an ∞-category (i.e., the horn-fillers may not be unique in the localization).

Note that I really mean we take the *category* of spaces, meaning I do not even define a topology on mapping spaces, and I only define the set of continuous maps. By localizing this ordinary category along weak homotopy equivalences, I can recover the ∞-category of spaces (which does see the homotopy groups of mapping spaces).

Remark VI.10.0.2. that the localization recovers the same ∞-category of Spaces as defined above is most efficiently proven using the theory of model categories, and Lurie's work in relating model categories to ∞-categories. See Proposition A.3.7.6, which states that presentable ∞-categories are the same thing as combinatorial model categories.

Example VI.10.0.3. One can make an ∞-category of left-bounded chain complexes by applying the dg-nerve to a dg-category spanned by a nice collection of chain complexes[15] or one could take the category of all left-bounded chain complexes and inverse quasi-equivalences. The two results are equivalent.[16] For unbounded chain complexes, see Section 1.3.5 of Higher Algebra; there, the result that "taking the nerve of a category of nice objects is the same thing as localizing the category of all objects" is Proposition 1.3.5.15.

[14] "Well-studied" is my way of gesturing at model categories.

[15] For details on this nice collection, see 1.3.2 of Higher Algebra. The dg nerve is later in this lecture.

[16] Theorem 1.3.4.4 of Higher Algebra, where the result is proven for general abelian categories satisfying an "enough projective objects" assumptions.

VI.11. (Not covered in spoken lecture) Localizations

Remark VI.11.0.1. Given any simplicial set $\mathcal{A}$, there is a natural way to make it into an ∞-category. Informally, each time you see a horn you cannot fill, you add on a simplex that fills it. One has to be rather careful doing this, but probably the one-line homotopy-theorist proof is that there is a model structure on all simplicial sets for which the ∞-categories are the fibrant objects.

Example VI.11.0.2 (Localization). Let $\mathcal{C}$ be an ∞-category. And let $W \subset \mathcal{C}$ be a subsimplicial set, which I will just assume is a collection of edge for this example. As in Remark VI.6.0.4, one can make a Kan complex $|W|$, which one can think of as a simplicial set obtained by turning a directed edge into a giant simplicial set with an edge going the other direction, 2-simplices indicating that the original directed edge and the new edge are homotpoy inverse to each other, 3-simplices indicating that the natural compositions of these edges do not give rise to any new data up to homotopy, et cetera.

So consider the collection of three simplicial sets, with two maps, drawn as follows:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathcal{C} \\ {|W|.} & \end{tikzcd}$$

Because simplicial sets consist of functors into the category of Sets, we can do anything in simplicial sets we can do for sets – in particular, we can glue simplicial sets together. So we can glue $\mathcal{C}$ and $|W|$ together along W. This produces a new simplicial set, but it may not have any horn-filling properties. By Remark VI.11.0.1, there is a natural way to make this simplicial set into an ∞-category.[17] We call this ∞-category

$$\mathcal{C}[W^{-1}]$$

and call it the *localization of* $\mathcal{C}$ *along* W. In a concrete way, this is the universal ∞-category obtained by inverting W in $\mathcal{C}$.

VI.12. (Not covered in spoken lecture) Limits and colimits inside an ∞-category

It can always seem incredibly vague when listening to somebody define what a co/limit ought to be in higher category theory – some universal thing that is initial/terminal up to some contractible choice, they say, and the hands wave aflutter. For a few decades, many students were just given models of homotopy limits/colimits for particular examples, and the models were natural enough that some version of the universal property certainly seemed to hold.

[17]If you are into model categories already, you can think of this new simplicial set as the homotopy pushout of the diagram.

Here we will define the notion of colimits in an ∞-category (without hand-waving). The notion of limits is dual – for example, one could take colimits in $\mathcal{C}^{\mathrm{op}}$, or define slice categories and initial objects in a natural way.

Remark VI.12.0.1. Note that I just say limit and colimit; I do not say homotopy limit and homotopy colimit. I could, but they would be synonyms in an ∞-category. Indeed, because there is no notion of "strict composition," one can't even define a notion of a (strict) limit/colimit.

The confusion often arises because something like the ∞-category of spaces is often thought of as starting with an actual category of spaces and then souping it up somehow; because of these strict origins, one sometimes likes to distinguish between classical constructions (which are often co/limits in a strict sense) and more homotopical constructions of homotopy co/limits.

In our lectures, we only ever constructed sequential colimits, and because all the maps in our sequences of spaces were nice inclusions, it turns out the classical/strict colimit, otherwise known as increasing union, models exactly the homotopy colimit.

Importantly, our Σ^∞-Ω^∞ adjunction was an adjunction of ∞-categories. In particular, Σ^∞ preserves (homotopy) colimits inside the ∞-category of spaces and of spectra.

Remark VI.12.0.2. When you see hand-waving by an expert in a talk or conversation – especially about limits and colimits – I assure you that the definitions are known. I bring up hand-waving not as a criticism of presentation, but an observation that we do sometimes have to remain vague for sake of time (and an acknowledge of how ∞-categories can seem to the non-user world). But now there is time to write and read, so I will write for you a version of what is already in Higher Topos theory, Section 1.2.13.

VI.12.1. Limits and colimits in categories. Fix a category $\mathcal{C}$. Fix another category $\mathcal{D}$, where we think of $\mathcal{D}$ as some "shape" that a diagram can be in.

Example VI.12.1.1. Let $\mathcal{D}$ be the category consisting of three objects $0, 1, 1'$ where there are exactly two non-identity morphisms: $0 \to 1$ and $0 \to 1'$. If you like, $\mathcal{D}$ is a poset with relations $0 \leq 1$ and $0 \leq 1'$ (with $1, 1'$ unrelated).

Then a functor from $\mathcal{D}$ to $\mathcal{C}$ is exactly the data of a diagram of the form

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \\ X_1' & \end{tikzcd}$$

inside $\mathcal{C}$. Here, X_0, X_1, X_1' are objects of $\mathcal{C}$ and the arrows are morphisms in $\mathcal{C}$. So if $\mathcal{C}$ were the category of abelian gropus, this is the data of three abelian groups and two group homomorphisms.

Now fix a functor $F : \mathcal{D} \to \mathcal{C}$. Given F, we can define a "slice" category as follows.

Definition VI.12.1.2 (The cone on $\mathcal{D}$; the slice category). Let $\mathcal{D}^{\triangleright}$ denote the category obtained from $\mathcal{D}$ by adjoining a new object – which I will call $*$ – and declaring that $\hom_{\mathcal{D}}(D, *)$ is a one-point set (for any object $D \in \operatorname{Ob} \mathcal{D}$). We call this the cone on $\mathcal{D}$.

We declare the *slice category*

$$\mathcal{C}_{\mathcal{D}/}, \qquad \text{or} \qquad \mathcal{C}_{F/}$$

(these are different notations for the same thing) to be the category of functors

$$G : \mathcal{D}^{\triangleright} \to \mathcal{C}$$

for which

$$G|_{\mathcal{D}} = F$$

where the lefthand side is the restriction of G to the subcategory $\mathcal{D} \subset \mathcal{D}^{\triangleright}$. So the slice category consists of functors G that extend F to one new point. A morphism in the slice category is a natural transformation from an G to another G' for which the natural transformation is the identity along $\mathcal{D}$.

Example VI.12.1.3. Let $\mathcal{D}$ be the diagram from Example VI.12.1.1. Then $\mathcal{D}^{\triangleright}$ is a category equivalent to the poset $[1] \times [1]$. And a functor from $\mathcal{D}^{\triangleright}$ to $\mathcal{C}$ is just a commutative square in $\mathcal{C}$.

In particular, if we fix a functor $F : \mathcal{D} \to \mathcal{C}$ and, a functor $G : \mathcal{D}^{\triangleleft} \to \mathcal{C}$ that extends F is simply a commutative square

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_1' \arrow[r] & W.
\end{tikzcd}$$

A morphism from G to G' is a diagram that commutes in $\mathcal{C}$, as follows:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \arrow[ddr] & \\
X_1' \arrow[r] \arrow[drr] & W \arrow[dr, "\eta"] & \\
& & W'
\end{tikzcd}$$

Given G and G', the arrow η above uniquely determines the map from G to G'. This is because of the definition of "commuting diagram," or the uniqueness of horn-fillers.

Definition VI.12.1.4. An object I of a category $\mathcal{S}$ is called *initial* if for every object $J \in \mathcal{S}$, we have that $\hom_{\mathcal{S}}(I, J)$ is a one-point set. In other words, I is called initial if it admits a unique morphism to any other object.

Remark VI.12.1.5. Initial objects are not unique, but they are unique up to unique isomorphism. You should make sure you understand what this means.

Definition VI.12.1.6. Fix a functor $F : \mathcal{D} \to \mathcal{C}$. A *colimit* for F is an initial object in $\mathcal{C}_{F/}$.

Example VI.12.1.7. In our working example, suppose G is a colimit – i.e., an initial object in $\mathcal{C}_{F/}$. In particular, W determines an object *equipped with a map* from X_0, X_1, X_1' making the relevant square commute. What the definition of initial object tells us is that if we have any other G' – i.e., any other object W' equipped with maps from the X_i, X_1' making the relevant square commute – then there exists a *unique* morphism η making the resulting triagnles commute:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \arrow[ddr] & \\
X_1' \arrow[r] \arrow[drr] & W \arrow[dr, "\exists ! \eta"] & \\
& & W'
\end{tikzcd}$$

If you have not see colimits before, the Exercise VI.27 will give you some examples.

Dually, one can define a slice category $\mathcal{C}_{/F}$ of objects mapping *to* the diagram given by F, and define a limit as a *terminal* category in $\mathcal{C}_{/F}$.

VI.12.2. Limits/colimits in ∞-categories. In classical category theory, we only need the two ideas of (i) slice categories, and (ii) terminal objects, to define the notion of colimit. So let's define these ideas for ∞-categories.

Definition VI.12.2.1. Given $\mathcal{D}$ an ∞-category, let $\mathcal{D}^{\triangleright}$ be the simplicial set obtained from $\mathcal{D}$ by adjoining a terminal vertex to $\mathcal{D}$.

Fix a map of simplicial sets $F : \mathcal{D} \to \mathcal{C}$ to an ∞-category $\mathcal{C}$. We define the *slice ∞-category*

$$\mathcal{C}_{F/}$$

to be the simplicial set whose k-simplices are as follows:

$$\text{Maps of simplicial sets } G : \mathcal{D}^{\triangleright} \times \Delta^k \to \mathcal{C} \text{ for which } G|_{\mathcal{D} \times \Delta^k} \equiv F.$$

In other words, we require that the restriction of G to $\mathcal{D} \times \Delta^k \subset \mathcal{D}^{\triangleright} \times \Delta^k$ to equal the composition $\mathcal{D} \times \Delta^k \to \mathcal{D} \xrightarrow{F} \mathcal{C}$.

EXAMPLE VI.12.2.2. In our working example, if $\mathcal{C}$ is an ∞-category, a functor $\mathcal{D}^{\triangleright} \to \mathcal{C}$ is now a diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_1' \arrow[r] & W \end{tikzcd}$$

in $\mathcal{C}$ – but this diagram has no notion of commuting on the nose! This combinatorial datum (of a functor $\mathcal{D}^{\triangleright} \to \mathcal{C}$) can be interpreted, healthily, as the data of the above morphisms, together with a particular *choice* of homotopy making the square commute up to homotopy.

DEFINITION VI.12.2.3. Let $\mathcal{S}$ be an ∞-category. We say that an object $I \in \mathcal{S}$ is *initial* if the mapping space[18] $\hom_{\mathcal{S}}(I, J)$ is contractible. (In this setting, this is equivalent to saying that the Kan complex $\hom_{\mathcal{S}}(I, J)$ has π_0 given by a one-element set, and has all $\pi_k = 0$ for $k \geq 1$.)

DEFINITION VI.12.2.4. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and fix a functor $F : \mathcal{D} \to \mathcal{C}$. Then a colimit for F is an initial object of $\mathcal{C}_{F/}$.

REMARK VI.12.2.5. Just as in the classical setting, a given functor $F : \mathcal{D} \to \mathcal{C}$ may not admit a colimit.

EXAMPLE VI.12.2.6. The most important example for us is that, in the ∞-category of pointed spaces, the pushout along the constant maps $X \to *$ is ΣX.

REMARK VI.12.2.7. Let $BG \to \mathcal{C}$ be a functor; i.e., an object of $\mathcal{C}$ with a homotopy coherent G action. limits and colimits of such functors compute homotopy fixed poin

VI.13. (Not covered in lecture) A_∞-categories versus ∞-categories

I was asked what the difference is between A_∞-categories and ∞-categories. For this, I have to be explicit who I'm talking to.

If you are a homotopy theorist through and through, you might live a life where you'd be surprised the two terms even exist. ∞-categories should be A_∞-categories, and vice versa.

But if you live in the communities converging at our conference, your vocabulary is different. A_∞-categories are, to you, absolutely some data involving objects, chain complexes, and formulas having to do with a cellular model of the Stasheff associahedra. On the other hand, ∞-categories are simplicial sets satisfying an inner-horn-filling condition.

So let us list some salient differences.

[18]See Exercise VI.22

(a) A_∞-categories are *linear*, or more precisely, linearly enriched. By this, I mean something very simple: Given two morphisms f, g in an A_∞-category (with same degree, and with domain and codomain) then they are elements of a chain complex. In particular, we know how to add them.

In contrast, ∞-categories come with no data on how to "add edges" with same domain and codomain.[19]

(b) A_∞-categories come with a specific composition. Namely, an m_2 operation. In contrast, ∞-categories only assert that one is specified with a family of triangles that could be interpreted as a homotopy-coherent diagram, but never privileges a particular triangle as exhibiting a composition.

(c) Likewise, A_∞-categories come equipped with specific homotopies realizing homotopy-associativity. As an example, that $dm_3 = m_2(m_2 \otimes \mathrm{id}) - m_2(\mathrm{id} \otimes m^2)$ is prescribed data of an m_3 that homotopes between the two natural associated products.[20] In contrast, ∞-categories do not privilege one tetrahedron over another as realizing an associativity up to homotopy.

[19]Instead, some ∞-categories have a *property* that naturally leads to such an addition; specifically, to a structure of spectrum on the mapping spaces. See Exercise VI.29.

[20]Note, by the way, that there could be plenty of other elements $H \in \hom^1$ for whom dH could equal $m_2(m_2 \otimes \mathrm{id}) - m_2(\mathrm{id} \otimes m_2)$. The A_∞ category, by defihnition, does not care about incorporating such H in its definition.

Exercises on simplicial sets and categories

VI.14. Simplicial relations

Verify that the (dual of) the simplicial relations (Remark VI.2.0.9) are satisfied by the morphisms σ_i and δ_i in Δ.

VI.15. The nerve of a category

For sake of notational sanity, we will let $N(\mathcal{C})$ denote the nerve of the category $\mathcal{C}$. So, for example, the set of k-simplices

$$N(\mathcal{C})_k$$

denotes the collection of commutative k-simplices in $\mathcal{C}$.

(a) Show that any functor $f : \mathcal{C} \to \mathcal{D}$ gives rise to a map of simplicial sets $N(\mathcal{C}) \to N(\mathcal{D})$.
(b) Conversely, show that any map of simplicial sets $N(\mathcal{C}) \to N(\mathcal{D})$ uniquely determines a functor $\mathcal{C} \to \mathcal{D}$.
(c) Show that the nerve operation N defines a fully faithful functor from the category of categories to the category of simplicial sets. (Note that the category of categories, in this exercise, does not have the data of natural transformations.)
(d) Show that a map of simplicial sets $\Delta^1 \times N(\mathcal{C}) \to N(\mathcal{D})$ is the same thing as a natural transformation.

VI.16. Categories using horn-fillers

Prove Theorem VI.5.0.1. You can ignore set issues if you like and consider only (small) simplicial sets and small categories.

VI.17. Maps from simplices are determined by their faces

(a) Any object $X \in \mathcal{C}$ defines a functor $\mathcal{C}^{\text{op}} \to \mathsf{Sets}$ by $\hom_{\mathcal{C}}(-, X)$. (This is the functor defining the Yoneda embedding.) This is the functor *represented by* X. Let $\mathcal{C} = \Delta$ and $X = [k]$. Show that the functor represented by $[k]$ is the nerve of the poset $[k]$. (We are using the fact that any poset can be considered a category here.)

Notation VI.17.0.1. We let Δ^k denote the simplicial set represented by $[k]$. We call it the *k-simplex.*

Warning VI.17.0.2. We are using Δ^k for both the topological simplex and the simplicial set. Sometimes, we will write $|\Delta^k|$ to denote the topological simplex when caution is necessary.

(b) Fix a simplicial set W. Using the Yoneda Lemma if you like, show that the set of k-simplices W_k is the same thing as the set of simplicial set maps $\Delta^k \to W$.

Definition VI.17.0.3. Let $\Delta_{\text{inj}} \subset \Delta$ be the (not full) subcategory with the same objects, but with only *injective* poset maps as morphisms. Then a functor $\Delta_{\text{inj}}^{\text{op}} \to \mathsf{Sets}$ is called a *semisimplicial set*. A map of semisimplicial sets is a natural transformation of such functors.

Note that any simplicial set gives rise to a semisimplicial set, simply by forgetting degeneracies.

(c) Going the other way, show that any semisimplicial set admits an "initial" simplicial set obtained by freely adjoining degeneracies. A convenient, but perhaps intimidating, way to phrase this construction is as follows: Given a functor $\Delta_{\text{inj}}^{\text{op}} \to \mathsf{Sets}$, you can left Kan extend this functor along the inclusion $\Delta_{\text{inj}}^{\text{op}} \to \Delta^{\text{op}}$.

Remark VI.17.0.4. While left Kan extensions can sound intimidating at first, I strongly encourage you to get used to them. They are incredibly powerful devices, and they are also universal in the following sense: left Kan extension is the left adjoint to restriction (e.g., left adjoint to the process of forgetting a simplicial set to a semisimplicial set). Right Kan extension is a right adjoint to restriction.

(d) Convince yourself that any not-necessarily-unital category gives rise to a nerve naturally interpretable as a semisimplicial set.
(e) Let $[k]'$ be the non-unital category obtained by discarding all identity morphisms from $[k]$. Show that $N([k])$ is the "free" simplicial set obtained from the semisimplicial set $[k']$.
(f) Convince yourself that (even though the k-simplex has infinitely many simplices) a k-simplex $\Delta^k \to X$ to a simplicial set X can be understood purely by what it does on the non-degenerate faces of Δ^k. (Note there are finitely many such faces in Δ^k.)

VI.18. Homotopy groups of Kan complexes

If Kan complexes are like spaces, there ought to be a definition for the homotopy groups of a Kan complex.

Definition VI.18.0.1. Let X be a Kan complex and $x_0 \in X_0$ a vertex. We define the quotient set

$$\pi_n(X, x_0) := \{\Delta^n \to X \text{ s.t. } \partial\Delta^n \equiv x_0\}/\sim$$

as follows.

First – before quotienting – we consider the set of all maps $\Delta^n \to X$ such that all the boundary faces of Δ^n are taken to the degenerate $(n-1)$-simplex given by x_0. (Pedantically, this degenerate simplex is given by $s_0 s_0 \dots s_0 x_0$.)

Now consider two functions $f, g : \Delta^n \to X$ satisfying the conditions of the previous paragraph. We define a *homotopy* form f to g to be a map

$$H : \Delta^{n+1} \to X$$

satisfying the following boundary conditions[21]:

- $d_0 H = f$
- $d_1 H = g$
- $d_i H \equiv x_0$ for all $i \geq 2$. (This is again lazy notation for saying that $d_i H$ is given by the degenerate simplex $s_0 \dots s_0 x_0$.)

We mod out by the equivalence relation of homotopy.

(a) Verify that when $X = \mathsf{Sing}(W)$ is the singular complex of a space W, a morphism $f : \Delta^n \to X$ with $\partial \Delta^n \equiv x_0$ can be interpreted as a map from a sphere to W based at x_0.
(b) Verify that homotopy is an equivalence relation.
(c) When $X = \mathsf{Sing}(W)$, verify that $\pi_0(X)$ – which you can define with the obvious modifications – is naturally in bijection with $\pi_0(W)$.
(d) For $k = 1$, and for any Kan complex X, verify that $\pi_1(X, x_0)$ is a group.
(e) For $k \geq 2$, and for any Kan complex X, verify that $\pi_2(X, x_0)$ is an abelian group.
(f) When $X = \mathsf{Sing}(W)$, it turns out that $\pi_n(X, x_0) \cong \pi_n(W, x_0)$ as abelian groups for all n and all x_0. Prove it.

VI.19. Simplicial groups

(a) Show that any simplicial group is a Kan complex.
(b) Let A be an abelian group. Construct a simplicial abelian group whose set of k-simplices is given by $A^{\times k}$ (so there is a unique 0-simplex) and where most of the face maps use the group addition.

VI.20. ∞-category basics: Homotopies between morphisms, and homotopy uniqueness of horn-fillers

Fix an ∞-category $\mathcal{C}$.

(a) Fix two edges $f, g \in \mathcal{C}_1$ with the same domain and codomain. (That is, $d_i f = d_i g$ for $i = 0, 1$.) We define a *homotopy* from f to g to be a 2-simplex

$$H : \Delta^2 \to \mathcal{C}$$

for which $d_0 H$ is degenerate, and $d_1 H = f, d_2 H = g$.

Fixing a domain and codomain vertex, show that the notion of homotopy between two morphisms is an equivalence relation.

[21] There are various riffs on this by swapping the role of d_0 and d_1 for other d_i and d_j.

Remark VI.20.0.1. One could have instead defined a homotopy to be a 2-simplex H for which d_2H is degenerate, and $d_0H = f$, $d_1H = g$.

Remark VI.20.0.2. More generally, given two k-simplices g and h with appropriate boundaries, one can say a *homotopy* between g and h is a $(k+1)$-simplex A with two boundaries given by g and h, and all other boundaries given by degenerate simplices. You should think about this when g and h are 2-simplices to give yourself a feel for how this works.

(b) Fix two edges f_{01}, f_{12} so that f_{ij} is an edge from an object x_i to an object x_j. Show that if g and h are $i = 1$th face of 2-simplices G and H for which

$$d_0H = d_0G = f_{12} \qquad \text{and} d_2H = d_2G = f_{01}$$

then g and h are homotopic.

Remark VI.20.0.3. This shows – by interpreting G and H as horn-fillers, and hence as putative compositions – that there is a well-defined composition up to homotopy.

In fact, one can show that the space of horn fillers is contractible. So putative compositions are in fact unique up to contractible choice of homotopy.

VI.21. From simplicial sets to spaces

The very definition of simplicial set was motivated as a prescription for taking some collection of simplices, and gluing boundary faces in a nice way. Thus there is a strong suggestion that any simplicial set should define, naturally, a topological space glued out of simplices.

Look up the definition of geometric realization $|X|$ of a simplicial set X. Convince yourself that there are natural maps

$$|\mathsf{Sing}(W)| \to W \qquad \text{and} \qquad X \to \mathsf{Sing}(|X|)$$

and that they induce isomorphisms on homotopy groups.

VI.22. ∞-category basics: Mapping spaces

Generalizing the previous exercise, let me explain we can construct a space (really, a Kan complex) of morphisms between any two objects.

Definition VI.22.0.1. Let $\mathcal{C}$ be an ∞-category, and fix two objects x and y. The (left) mapping space of maps from x to y is defined as a simplicial set whose k-simplices are

$$\{a : \Delta^{k+1} \to \mathcal{C} \qquad | \qquad a|_{\{0\}} = x, \text{ and } a|_{d_0\Delta^{k+1}} \equiv y\}.$$

In other words, a k-simplex in the mapping space is a $(k+1)$-simplex in $\mathcal{C}$ whose 0th vertex is sent to x, and whose 0th face is the degenerate simplex at y. We will denote this Kan complex by

$$\hom_{\mathcal{C}}(x, y).$$

REMARK VI.22.0.2. This simplicial set is a Kan complex (hence deserves to be thought of as a space) – this is Exercise **??**.

REMARK VI.22.0.3. There is another model for mapping spaces – by taking those maps $\Delta^{k+1} \to \mathcal{C}$ for which the $(k+1)$st face is degenerate at x, and those $k+1$st vertex has value y. It is a theorem that these two mapping spaces are homotopy equivalent. See 1.2.2 of Higher Topos Theory.

(a) When $\mathcal{C}$ is an ∞-category, Show that $\hom_{\mathcal{C}}(x, y)$ is a Kan complex.
(b) Let C be a topologically enriched category. Is there a natural map between the mapping space from x to y in $N(C)$, and the space of morphisms in C from x to y?
(c) Show that if $f : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, there is an induced map of Kan complexes $\hom_{\mathcal{C}}(x, y) \to \hom_{\mathcal{D}}(f(x), f(y))$.

VI.23. ∞-category basics: Functor categories

Because a simplicial set is just a functor to sets, all the nice properties about the category of sets are inherited. (This follows the basic math principles that "the collection of maps into BLAH" inherits the structure of BLAH.)

In particular, given two simplicial sets X and Y, we define their *product* simplicial set to have set of k-simplices given by

$$(X \times Y)_k = X_k \times Y_k.$$

I leave it to you to figure out the face and degeneracy maps, or more generally, what to do on morphisms $[k] \to [k']$. Likewise, there are limits, colimits, et cetera, of simplicial sets, computed level-wise. (The colimit of a diagram X_α of simplicial sets is computed by declaring the k-simplex set to be the colimit of the induced diagram $(X_\alpha)_k$ of sets.)

DEFINITION VI.23.0.1. Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets. We will define

$$\mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

to be a new simplicial set, whose k-simplices are given by

$$\hom_{\mathrm{sSets}}(\mathcal{C} \times \Delta^k, \mathcal{D}).$$

Here we'll see how useful the product is.

(a) Convince yourself that if $\mathcal{D}$ is an ∞-category, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
(b) Now let $\mathcal{C}$ also be an ∞-category. Explain the sense in which a 0-simplex of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is a functor from $\mathcal{C}$ to $\mathcal{D}$. Explain the sense in which a 1-simplex is a natural transformation. Explain the sense in which a 2-simplex is homotopy-coherent diagram of natural transformations.

REMARK VI.23.0.2. Many models of higher category theory lack the ease with which one can "enrich" over itself. The above is one very appealing feature of the model of ∞-categories. The other compelling feature is the theory of fibrations, which I hope to get to.

VI.24. ∞-category basics: Equivalences in an ∞-category

Definition VI.24.0.1. Let $\mathcal{C}$ be an ∞-category and fix an edge $f : x \to y$ in $\mathcal{C}$. We say that f is an *equivalence* if there exists an edge $g : y \to x$ and two 2-simplices A, B satisfying

$$d_1 A = s_0 x, \qquad d_2 A = f, \qquad d_0 A = g,$$

and

$$d_0 B = s_0 y, \qquad d_0 B = f, \qquad d_2 B = g.$$

(a) Convince yourself that the above definition is telling you that f is a morphism which is invertible up to homotopy.
(b) You may wonder about the data of g, A, B. Show that any other data g', A', B' exhibiting f as an equivalence is homotopic to g, A, B. (See Remark VI.20.0.2.) In particular, any two homotopy inverses to f are homotopic.

Remark VI.24.0.2. More generally, one can show that the space of (g, A, B) exhibiting homotopy inverses to f is a contractible space.

(c) Using the ∞-category of ∞-categories and this exercise, write down what you would mean by an equivalence of ∞-categories.
(d) It is a theorem that equivalences of ∞-categories can be detected algebraically. Put another way, suppose you have a functor $f : \mathcal{C} \to \mathcal{D}$ such that
  (i) f is essentially surjective. This means that every object of $\mathcal{D}$ is equivalent (in $\mathcal{D}$) to an object in the image of f.
  (ii) f is fully faithful. This means that for every pair of objects $x, y \in \operatorname{Ob}\mathcal{C}$, the map

$$\pi_0 \hom_{\mathcal{C}}(x, y) \to \pi_0 \hom_{\mathcal{D}}(f(x), f(y))$$

  is a bijection, and for every choice of connected component of $\hom_{\mathcal{C}}(x, y)$, the induced map

$$\pi_k \hom_{\mathcal{C}}(x, y) \to \pi_k \hom_{\mathcal{D}}(f(x), f(y))$$

  is a bijection for all $k \geq 1$.

Prove that if f is an equivalence in the sense of your previous solution, then f is essentially surjective and fully faithful.
(e) It turns out that any functor between ∞-categories that is essentially surjective and fully faithful is, in fact, an equivalence of ∞-categories. (The easiest proof I know of this would use a model category structure on simplicial sets where the fibrant objects are simplicial sets.) There is a non-categorical version of this fact. Convince yourself of, or read a proof of, Whitehead's theorem: If a map between CW complexes induces an isomorphism on all homotopy groups, then it admits a homotopy inverse.

VI.25. The homotopy coherent nerve

We now exposit Example VI.8.0.5. Further reading is Section 1.1.5 of Higher Topos Theory.

NOTATION VI.25.0.1 (The fattened simplex category). Let $I \cong [n]$ be a linearly ordered, non-empty, finite set. Given two elements $i \leq j \in I$, we let

$$P_{i,j}$$

consist of those subsets $A \subset I$ that contain i and j, and only elements between i and j.

Then $P_{i,j}$ is a poset under inclusion of subsets, hence a category. We let $N(P_{i,j})$ be the nerve.

We note there are maps of posets

$$P_{i,j} \times P_{j,k} \to P_{i,k}, \qquad (A,B) \mapsto A \bigcup B$$

and hence maps

$$N(P_{i,j}) \times N(P_{j,k}) \to N(P_{i,k}).$$

We let

$$\mathfrak{C}[\Delta^n]$$

denote the category enriched in simplicial sets with objects $i \in [n]$, and

$$\hom(i,j) := N(P_{i,j}).$$

(a) Let C be a category enriched in topological spaces. By applying Sing to every morphism space, obtain a category C' enriched in simplicial sets.

 DEFINITION VI.25.0.2. The homotopy coherent nerve of C is defined to be the simplicial set

$$N(C)_n := \hom(\mathfrak{C}[\Delta^n], C')$$

 where the set of n-simplices is given by the collection of functors (of simplicially enriched categories) from the fact n-simplex category to C'.

(b) Recover the descriptions of $N(C)_n$ for $n = 0, 1, 2$ in Example VI.8.0.5.
(c) Draw a picture describing what a 3-simplex of $N(C)$ is.
(d) Show that a functor $C \to D$ of Top-enriched categories gives rise to a functor $N(C) \to N(D)$ of ∞-categories.
(e) Exhibit examples of C and D and a functor $N(C) \to N(D)$ that does not arise from a Top-enriched functor $C \to D$.

VI.26. The dg- and A_∞-nerves

(a) Look up the dg nerve construction in Lurie's Higher Algebra[22] Construction 1.3.1.6.
(b) Verify the dg-nerve is an ∞-category.

[22] `https://www.math.ias.edu/~lurie/papers/HA.pdf`

(c) Let $\mathcal{A}$ be a dg-category and fix two objects x, y. Verify that π_k of the morphism space in $N(\mathcal{A})$ from x to y is isomorphic to the homology groups $H_k \hom_{\mathcal{A}}(x, y)$ for $k \geq 0$.
(d) Verify that a functor of dg-categories gives rise to a functor between their dg-nerves.
(e) Look up in either Faonte[23] or Tanaka's[24] works the definition of an A_∞-nerve of an A_∞-category. Verify that the A_∞-nerve of an A_∞-category is an ∞-category.

VI.27. Colimits, classically

Here are some examples of colimits in some common classical settings. These examples are incredibly important, no matter how ∞-categorical you want to get.

(a) Suppose $\mathcal{D} = \mathbb{Z}$ is the poset of integers. A functor $\mathcal{D} \to \mathcal{C}$ is thus the data of objects $X_i, i \in \mathbb{Z}$ and maps $X_i \to X_{i+1}$. When $\mathcal{C} = \mathsf{Sets}$, show that a colimit for this diagram is (in bijection with) the increasing union $\bigcup_i X_i$ (glued along the maps f_i).
(b) Suppose that $\mathcal{D} = [1] \times [1] \setminus \{(1,1)\}$ is the diagram from Example VI.12.1.1. A colimit for a functor $\mathcal{D} \to \mathcal{C}$ is called a *pushout*. Show that the pushout for $\mathcal{C} = \mathsf{Sets}$ is given by the set $X_1 \bigcup_{X_0} X_1'$ (following the notation from Example VI.12.1.1).
(c) Let $\mathcal{C} = \mathsf{Ab}$ be the category of abelian groups. Show that the pushout is given by the quotient of $X_1 \oplus X_1'$ by the image of the embedding $X_0 \to X_1 \oplus X_1'$.
(d) Suppose $\mathcal{D}$ is a category with two objects and only identity morphisms (so the two objects have no morphisms between each other). A functor $\mathcal{D} \to \mathcal{C}$ is just the data of two objects X_1, X_2. Show that if $\mathcal{C} = \mathsf{Sets}$, the colimit is $X_1 \coprod X_2$ (disjoint union of sets). Show that if $\mathcal{C} = \mathsf{Ab}$, the colimit is $X_1 \oplus X_2$ (the direct sum).

VI.28. Homotopy colimit basics

For the notation here, I refer you to Section VI.12.2.

(a) Let $\mathcal{D}$ be a simplicial set. Come up with a definition of $\mathcal{D}^{\triangleright}$. (As a hint, your definition should have a map of simplicial sets down to Δ^1, where the fiber above 0 is $\mathcal{D}$ and the fiber above 1 is Δ^0. As another hint, your answer should recover – when $\mathcal{D}$ is the nerve of an ordinary category – the nerve of the classical $cD^{\triangleright}$.)
(b) Convince yourself that $\mathcal{C}_{F/}$ is an ∞-category.

[23] Giovanni Faonte, "Simplicial nerve of an A-infinity category." Theory and Applications of Categories, Vol. 32, No. 2, 2017, pp. 31–52.

[24] Hiro Lee Tanaka, "The Fukaya category pairs with Lagrangian cobordisms." arXiv:1607.04976.

(c) Fix a pointed topological space X (a CW complex if you like), and consider the diagram given by the continuous maps $* \leftarrow X \rightarrow *$. Convince yourself that the colimit of this diagram in the ∞-category of spaces (i.e., the pushout in the ∞-categorical sense, i.e., the homotopy pushout in a colloquial sense) is homotopy equivalent to the reduced suspension of X. (Hint: What does it mean to give two maps from $X \rightarrow W'$ that factors through a point, and to give a homotopy between two such maps?)
(d) In contrast, show that the ordinary colimit of $* \leftarrow X \rightarrow *$ is just a point.

VI.29. Stable ∞-categories

Another big application of ∞-categories is a resolution to an age-old problem: That Verdier's notion of "triangulated category" is clunky and not suitable for many applications. For example, if you want to prove the intuitive fact that D^bCoh of variaties should glue when you glue varieties together, you quickly realize that you need more data to glue complexes (the objects) together and to characterized the glued-together complexes in a useful way.

The notion of a stable ∞-category[25] is one in which these issues are solved for the following reason:

(1) Stability is a property, not extra structure. [26]
(2) The ∞-category of stable ∞-categories has all limits and colimits; gluing these ∞-categories can hence be phrased as computing an appropriate limit of ∞-categories. In contrast, the (ordinary[27]) category of triangulated categories lacks this property.

Definition VI.29.0.1. An ∞-category $\mathcal{C}$ is called *stable* if

(1) $\mathcal{C}$ has an object that is both initial and stable – i.e., a zero object.
(2) $\mathcal{C}$ admits all finite limits and finite colimits.
(3) A diagram $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$ is a colimit diagram (i.e., a pushout diagram) if and only if it is a limit diagram (i.e., a pullback diagram). In other words, pushout diagrams are pullback diagrams.[28]

[25]The canonical reference is Chapter 1 of Lurie's Higher Algebra. I highly recommend perusing it.

[26]In contrast, a triangulated category has some structures satisfying some axioms. The main difference is that stability is defined via limits/colimits, which are ideas that only involve testing against yes/no questions. But a triangulated structure more or less forces your category to forget limits/colimits, and instead label the diagrams you would want to be limits/colimits, while forgetting the homotopical structures that define limits/colimits to begin with!

[27]Because triangulated structures lack homotopical structures, there is no natural notion of an ∞-category of triangulated categories. Indeed, the modern analogue of such a thing is the ∞-category of stable ∞-categories.

[28]This last axiom is by far the most important feature of stability.

EXAMPLE VI.29.0.2. The most basic pushout one could take is the pushout along the zero morphisms

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

or pullback along the zero morphisms

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & X. \end{tikzcd}$$

(To even draw these diagrams, we are using the property that zero objects exist – we are using $*$ to denote the zero object). Let us denote the pullback by $X[-1]$ and the pushout by $X[1]$. Then the axiom of stability implies that $X[1][-1] \simeq X$ – in other words, the operation of sending an object X to $X[1]$ (defined via pushouts along the zero map!) is an invertible operation with inverse $[-1]$.

(a) In the ∞-category of pointed spaces, show that the pullback/pushout of the diagrams in Example VI.29.0.2 recover ΩX and ΣX. Find an example of a pointed space for which $\Omega\Sigma X$, or $\Sigma\Omega X$, is not homotopy equivalent to X. Conclude that the ∞-category of pointed spaces is not stable.
(b) Convince yourself that for any ∞-category $\mathcal{C}$, and for any diagram $F : \mathcal{D} \to \mathcal{D}$, and for any object Y of $\mathcal{C}$, there is a natural map of spaces (really, Kan complexes)
$$\hom_{\mathcal{C}}(W, \lim_{d \in \mathcal{D}} F(d)) \to \lim_{d \in \mathcal{D}} \hom_{\mathcal{C}}(W, F(d))$$
which is an equivalence.
(c) Conclude that $X[-1]$ represents the functor $\Omega \hom_{\mathcal{C}}(, -X)$.
(d) Conclude that for any stable ∞-category $\mathcal{C}$ and any two objects $W, X \in \mathcal{C}$, the space $\hom_{\mathcal{C}}(X, Y)$ can be given the structure of a spectrum.[29]
(e) It turns out that the last axiom of stability (that pushouts are pullbacks) is equivalent to the axiom that the functor $X \mapsto X[-1]$ is invertible.[30] Using this fact, convince yourself that the ∞-category of spectra is stable.
(f) Look up the definition of triangulated category. Convince yourself that if we replace a stable ∞-category $\mathcal{C}$ with a category $ho\mathcal{C}$with the same set of objects, but where $\hom_{ho\mathcal{C}}(x, y) = \pi_0 \hom_{\mathcal{C}}(x, y)$, then $ho\mathcal{C}$ is a triangulated category. See also Section 1.1.2 of Higher Algebra.

[29]In fact, any stable ∞-category is enriched in spectra – the hard part here is defining what one means by enrichment.

[30]Corollary 1.4.2.27 of Higher Algebra.

LECTURE VII

Fibrations of ∞-categories and symmetric monoidal ∞-categories

Be running up that road
Be running up that hill
Be running up that building
– Kate Bush[1]

Last time we saw some advantages of ∞-categories:

(1) ∞-categories are presented by a concrete combinatorial theory. The definition of functor is also combinatorial and concrete.
(2) ∞-categories allow us to generalize all classical categorical ideas like limits and colimits. Moreover, the notion of functor allows us to speak of strictly commuting diagrams and homotopy coherent diagrams at the same time.
(3) There are concrete ways to talk about localizations, and about functor ∞-categories. It is also easy to construct the ∞-category of ∞-categories.

If none of these seem like advantages to you, that's okay. Like so many technological advances in society, the utility is only clear if you have occasion to use the technology.

Today we will try to see another huge advantage: ∞-categories have a theory of fibrations.

Fibrations allow us to make incredibly useful constructions – by looking at pullbacks, at sections, et cetera – of important ideas. Moreover, just as "a family of spaces indexed by a base space B" is the same as a fibration or a bundle over B (in the classical theory of spaces) we'll see that "a family of categories indexed by a base category $\mathcal{C}$" – otherwise known as a functor from the ∞-category $\mathcal{B}$ to the ∞-category of ∞-categories – can be modeled by the notion of a *coCartesian fibratoin.*

[1] There is no particular meaning I have in mind for this quote; I was just listening to this song during my writing this week. You can blame Stranger Things.

One amazing application of fibrations is that fibrations allow us to give non-trivial characterizations of limits and colimits in the ∞-category of ∞-categories. I will not get to that in spoken lecture, but I hope to write up something later for your perusal.

If time allows, we will arrive also at the definition of ∞-operads and symmetric monoidal ∞-categories. I hope to give you a very convincing definition of a commutative algebra in a symmetric monoidal ∞-category; this recovers what one would normally call an E_∞-algebra. Secretly, the hardest part here is motivating a category $\mathcal{F}\mathsf{in}_*$ of finite pointed sets.

Of course, just getting to a definition can seem anticlimactic. But I will make things less anticlimactic – by giving another definition (paradoxically). Given a symmetric monoidal ∞-category, a commutative algebra in it (or, equivalently, an E_∞-algebra in it) is simply a nice section of a coCartesian fibration. Some formal conclusions are immediate from the definitions.

Remark VII.0.0.1. I must admit that, in giving this last lecture, I am starting to have in mind the student who has been reading, or wants to read, some of Jacob Lurie's writings. I want to clear some of the confusions that might arise in reading such tomes alone (as I did often in my graduate days), so that these notes at least might streamline the process, and provide a companion to get you started on your journey.

And, to be honest, Lurie does an excellent job – at the start of each chapter and section – to motivate what the overarching goals are. So my lectures may not even contribute much! I hope you enjoy anyway.

VII.1. What is a symmetric monoidal ∞-category?

What should a symmetric monoidal ∞-category be? (If you aren't familiar with the classical notion of symmetric monoidal category, see Remark II.2.0.1.)

Roughly speaking, a symmetric monoidal category is like a commutative monoid (or, commutative algebra) in categories. So let's review what being a "commutative monoid" is supposed to entail. It should consist of the data of a single entity A, along with maps

(1) $m : A \times A \to A$ (the product),
(2) $1 : * \to A$ (the unit map),

and (classically) these data should satisfy some *properties*:

(a) A symmetric equivariance. That is, the diagram

(VII.1.1)
$$\begin{tikzcd} A \times A \arrow[rr, "\text{swap}"] \arrow[dr] & & A \times A \arrow[dl] \\ & A & \end{tikzcd}$$

should commute.

(b) A *unitality* condition. Namely, the diagrams

$$\begin{tikzcd} A \simeq * \times A \arrow[r, "1\times \mathrm{id}_A"] \arrow[dr, "\mathrm{id}_A"'] & A \times A \arrow[d] & A \times * \cong A \arrow[l, "\mathrm{id}_A \times 1"'] \arrow[dl, "\mathrm{id}_A"] \\ & A & \end{tikzcd}$$

(c) An *associativity* condition. Namely, the diagram

$$\begin{tikzcd} A \times A \times A \arrow[r, "\mathrm{id}_A \times m"] \arrow[d, "m\times \mathrm{id}_A"'] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"] & A \end{tikzcd}$$

should commute.

You are by now used to the philosophy that, if I want some homotopical or higher-categorical notion of a commutative monoid, I ought to replace each of the diagrams above by some *homotopy-commuting* diagrams, so that (for example) m is associative up to homotopy. And, we ought to specify those homotopies – just as A_∞-categories provide homotopies, and just as $\mathbb{E}_n$-algebras are provided with maps from the spaces $\mathbb{E}_n(k)$ to spell out the precise homotopies realizing the intuitions of "commutative up to dimension $n-1$."

At the same time, we quickly see that – just as we needed to consider infinitely many horns – there is an infinite amount of data one would need to track – if two elements at a time commute, what if we need to understand how three elements at a time commute? Or N?

We begin writing down what we mean, but then we fear that (unless we are clever about how to organize all the higher data) we enter hand-waving territory. That would be bad for proving anything. So the key insight will be, again, to nail down the combinatorics of what it means to be a commutative algebra up to coherent homotopy. Let's do this.

The resultant combinatorics will be incredibly simple and satisfying – we just remove orderings from the things that organzied homotopy associativity. That is, we remove the orders from the objects of Δ. And, for the sake of unitality, we will also allow for the empty set. (You'll see what I mean.)

VII.1.1. Finite sets, and functors out of their category.

Definition VII.1.1.1. We let $\mathcal{F}\mathsf{in}$ denote the category of finite sets (with morphisms given by functions – yes, good old functions).

I know it seems quite silly to say "Hey, let's explore the category of finite sets with functions between finite sets." I mean, aren't we all familiar with finite sets? We are, but for many of us, we may not have really thought of the combinatorial structure they encode. So let's explore.

Notation VII.1.1.2. Let $\mathcal{C}$ be an ∞-category, and fix a functor $Z : \mathcal{F}\mathsf{in} \to \mathcal{C}$.

Assumption VII.1.1.3 (An assumption we will see how to make more natural). We will make a major assumption: That $\mathcal{C}$ has some notion[2] of "direct product" and that for a finite set I and an object $A \in \mathcal{C}$, we have an identification

$$Z(I) \cong A^I.$$

That is, $Z(I)$ is the I-fold product of a fixed object I.

Remark VII.1.1.4. Note that when $I = \emptyset$, $A^\emptyset$ is some object universal for maps into 0 copies of A — that is, given no data, any category has a unique map to $A^\emptyset$. Thus $Z(\emptyset)$ must be a terminal object.

Because any object I is in bijection with the set $\underline{n}$ with elements $\{1, \ldots, n\}$, I will be slightly more concrete and only write out what such a functor Z encodes on objects of the form $\underline{n}$.

Example VII.1.1.5 ($n = 0$). Let $\underline{0} = \emptyset$. Then there are unique morphisms $\underline{0} \to \underline{n}$ for any n, and in particular, we have a map

$$Z(\underline{0}) \to Z(\underline{1})$$

i.e., a map

$$* \to A$$

where we let $*$ denote the empty product – an initial object in our case. We will find that this is a unit map in Example VII.1.1.8.

Example VII.1.1.6. For any n, there is a unique map $\underline{n} \to \underline{1}$. In particular, the $n = 2$ case specifies a single morphism

$$m : A \times A \to A.$$

Example VII.1.1.7 (The start of commutativity). Now, for $n = 2$, the set $\underline{2}$ has a natural automorphism – swap. And the swap map fits into two interesting diagrams in $\mathcal{F}\mathsf{in}$:

$$\begin{tikzcd} \underline{2} \arrow[rr, "\text{swap}"] \arrow[dr] & & \underline{2} \arrow[dl] \\ & \underline{1} & \end{tikzcd}, \qquad \begin{tikzcd} \underline{2} \arrow[rr, "\text{swap}"] \arrow[dr, "\mathrm{id}_{\underline{2}}"'] & & \underline{2} \arrow[dl, "\text{swap}"] \\ & \underline{2} & \end{tikzcd}.$$

The second map says, of course, that the swap map is of order 2. Now, a functor $\mathcal{F}\mathsf{in} \to \mathcal{C}$ (with Assumption VII.1.1.3), sends the above 2-simplices to the following 2-simplices in $\mathcal{C}$:

(VII.1.2)
$$\begin{tikzcd} A^2 \arrow[rr, "Z(\text{swap})"] \arrow[dr, "m"'] & & A^2 \arrow[dl, "m"] \\ & A & \end{tikzcd}, \qquad \begin{tikzcd} A^2 \arrow[rr, "Z(\text{swap})"] \arrow[dr, "\mathrm{id}_{A^2}"'] & & A^2 \arrow[dl, "Z(\text{swap})"] \\ & A^2 & \end{tikzcd}.$$

---

[2]Really, we will assume that $\mathcal{C}$ has finite products, in the categorical sense. We talked last time about how to talk about limits, and products are limits of diagrams given by finite, discrete sets. The example of interest to us will be $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$, the ∞-category of ∞-categories, which does admit products (given by the usual product of simplicial sets).

Here, the fact that the edges (i.e., faces) of the triangles are as indicated follows from the fact that a simplicial set map is compatible with face maps. The notation "id_{A^2}" means $s_0 A^2$; the 0th degeneracy maps. That we know the edges labeled as such are degeneracy maps is a consequence of the fact that a simplicial set map is compatible with degeneracy maps.

Let's stare at (VII.1.3). We are so in business! The left triangle in (VII.1.3) precisely tells us that m, and something that deserves to be called m precomposed with swap, are the same map up to a homotopy specified by the triangle. This triangle should of course be compared with (VII.1.1).

Likewise, the righthand triangle in (VII.1.3) tells us that the swap map is idempotent – this is an important condition one sees when defining, for example, symmetric monoidal categories classically (where $\mathcal{C} = \mathsf{Cat}$ and A is a category).

But there is so much more. Why stop with $n = 2$? There are unique maps $\underline{n} \to \underline{1}$. Z of this map should be thought of as "a single, specified way to take n elements in A and multiply them together." We will explore this single way in a moment, but let's for now observe that it is very homotopy-commutative-looking. Namely, take any automorphism (i.e., permutation) $\sigma : \underline{n} \to \underline{n}$. Clearly σ fits into the following commutative diagrams in $\mathcal{F}\mathsf{in}$:

$$\begin{tikzcd} \underline{n} \arrow[rr, "\sigma"] \arrow[dr] & & \underline{n} \arrow[dl] \\ & \underline{1} & \end{tikzcd}, \qquad \begin{tikzcd} \underline{n} \arrow[rr, "\sigma"] \arrow[dr, "\mathrm{id}_{\underline{n}}"'] & & \underline{n} \arrow[dl, "\sigma^{-1}"] \\ & \underline{n} & \end{tikzcd}.$$

(We have denoted the inverse to σ by σ^{-1}.) A functor $\mathcal{F}\mathsf{in} \to \mathcal{C}$ (with Assumption VII.1.1.3), sends the above 2-simplices to the following 2-simplices in $\mathcal{C}$:

$$\begin{tikzcd} A^n \arrow[rr, "Z(\sigma)"] \arrow[dr, "m"'] & & A^n \arrow[dl, "m"] \\ & A & \end{tikzcd}, \qquad \begin{tikzcd} A^n \arrow[rr, "Z(\sigma)"] \arrow[dr, "\mathrm{id}_{A^n}"'] & & A^n \arrow[dl, "Z(\sigma^{-1})"] \\ & A^n & \end{tikzcd}. \tag{VII.1.3}$$

The lefthand triangle says that regardless of how we permute the input factors of m, we have a specified triangle showing that the permuted product is homotopic to the given n-fold product. The righthand triangle is the beginnings of a symmetric group action on A^n; the diagram itself only illustrates that $Z(\sigma^{-1})$ is an inverse to $Z(\sigma)$, up to a homotopy specified by the given triangle.

Example VII.1.1.8 (Unitality). In $\mathcal{F}\mathsf{in}$, the identity map of $\underline{1}$ factors in two ways.

$$\begin{tikzcd} \underline{1} \arrow[r] \arrow[dr, "\mathrm{id}"'] & \underline{2} \arrow[d] & \underline{1} \arrow[l] \arrow[dl, "\mathrm{id}"] \\ & \underline{1} & \end{tikzcd}$$

where the horizontal arrows are the two ways to map $\underline{1}$ into $\underline{2}$ – missing the element 1, or missing the element 2. Applying Z to the above diagram and again under Assumption VII.1.1.3,

$$\begin{tikzcd} A \simeq * \times A \arrow[r, "1\times \mathrm{id}_A"] \arrow[dr, "\mathrm{id}_A"'] & A \times A \arrow[d, "m"] & A \times * \cong A \,. \arrow[l, "\mathrm{id}_A \times 1"'] \arrow[dl, "\mathrm{id}_A"] \\ & A & \end{tikzcd}$$

I think, by now, you don't need me to explain the sense in which these triangles exhibit that 1 is indeed a unit map up to homotopy. You can also explore what homotopies this diagram enjoys when swapping the single copy of $A \times A$ in the diagram.

Example VII.1.1.9 (Associativity and commutativity). I leave it to you to explore the associativity encoded in $\mathcal{F}\mathsf{in}$. For example, how can we factor the single map $\underline{3} \to \underline{1}$? How do these factorizations play with the symmetric group actions?

VII.1.2. A first guess. I hope that I've started to convince you that the combinatorics of (the category of) finite sets has, amazingly, encoded what it means to be a commutative algebra all along. This is an "obvious" fact once you think of just how much commutativity has to do with permutations, but is certainly hidden from us in most classes of algebra. This is perhaps an artifact of the truth that – without higher homotopies – one rarely needs to contemplate finite sets of cardinality larger than 3. (After all, even for multiplying 17 elements together, the strict notions of commutativity and associativity allows us to worry only about parenthesizing 3 elements and permuting 2 elements at a time.)

Let's now take $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$ to be the ∞-category of ∞-categories.

Definition VII.1.2.1 (First attempt). A symmetric monoidal ∞-category is a functor

$$Z : \mathcal{F}\mathsf{in} \to \mathcal{C}\mathrm{at}_\infty$$

that sends any object $I \in \mathcal{F}\mathsf{in}$ to the product $Z(\underline{1})^I$.

This is a perfectly fine definition, and is probably the most intuitive definition you could have. But the requirement that $Z(I)$ *equal* the I-fold product of $Z(\underline{1})$ – or even the data of such an identification – is clunky to carry around. Indeed, our "first attempt" definition is unnatural in that we should also demand that $Z(I)$'s action but $\mathrm{Aut}(I)$ match the action of $\mathrm{Aut}(I)$ on $Z(\underline{1})^I$. This will be a lot to "require" or a lot to encode using additional data.

Thankfully, there is a notion of (i) *coCartesian fibration* that will help us in the long run – once we also incorporate (ii) the combinatorics of adjoining basepoints to our finite sets. This lecture is more packed than the others because we need to explicate these *two* unfamiliar ideas for the audience.

VII.2. Grothendieck constructions and fibrations

I will first start talking about the classical notion – an idea of fibrations for categories.

Suppose you have a continuous function $f : X_0 \to X_1$. We actually have three equivalent ways to think about f:

1. As a continuous function $f : X_0 \to X_1$.
2. As a functor from the category with two objects (called 0 and 1) and a single non-identity morphism (from 0 to 1) to the category of spaces.
3. As a a space C_f (the mapping cylinder of f) equipped with a continuous map

$$\begin{tikzcd} C_f \arrow[d, "p"] \\ \Delta^1 \end{tikzcd}$$

to the 1-simplex Δ^1, whose fiber above the initial vertex of Δ^1 is X_0, whose fiber above the terminal vertex of Δ^1 is given by X_1, and which satisfies a directed fibration property (which we do not make precise here).

Picture of mapping cylinder

This has an analogue in the theory of categories. The following data all turn out to be equivalent:

1. Two categories X_0 and X_1, along with a functor $f : X_0 \to X_1$.
2. A functor from the category [1] with two objects (called 0 and 1) and a single non-identity morphism (from 0 to 1) to the category of categories.
3. A category $\mathcal{C}_f$ (the Grothendieck construction of f) equipped with a functor map

$$\begin{tikzcd} \mathcal{C}_f \arrow[d, "p"] \\ \Delta^1 \end{tikzcd}$$

to the category Δ^1 (which is the domain category appearing in (2)), whose fiber above the initial vertex of Δ^1 is X_0, and whose fiber above the terminal vertex of Δ^1 is given by X_1. p must satisfy a fibration property we articulate in Section VII.4.

In case you haven't see this sort of thing before, let me say that it's a "thing" in topology to try to replace certain complicated data with the data of a single fibration-like map p over some base. This has the advantage of allowing us to articulate compatibility of certain data as maps between fibrations.

Example VII.2.0.1. The data of a continuous G-action on a space X gives rise to the data of a fibration $p : E \to BG$ whose fibers are X, and where monodromies in BG precisely articulate the way in which G acts on X. Indeed, *homotopy coherent* actions of G on X are precisely equivalent to fibrations over BG with fiber X. Then a map $E \to E'$ respecting p and p' is the same thing as a map $X \to X'$ equipped with data exhibiting G-equivariance up to homotopy.

Construction VII.2.0.2 (Grothendieck construction). Let me tell you how the Grothendieck construction $\mathcal{C}_f$ is defined. Fix a functor $f : X_0 \to X_1$. Then $\mathcal{C}_f$ is a category whose set of objects is $\operatorname{Ob} X_0 \coprod \operatorname{Ob} X_1$. Given two objects $x, y \in \mathcal{C}_f$, we declare

$$\hom_{\mathcal{C}_f}(x, y) := \begin{cases} \hom_{X_0}(x, y) & x, y \in X_0 \\ \hom_{X_1}(x, y) & x, y \in X_1 \\ \emptyset & x \in X_1, y \in X_0 \\ \hom_{X_1}(f(x), y) & x \in X_0, y \in X_1. \end{cases}$$

One has an obvious functor from $\mathcal{C}_f$ to Δ^1.

More generally, if $f : B \to \mathsf{Cat}$ is a functor from a category B to the category of categories, there is a natural functor $p : \mathcal{C}_f \to B$ one can define analogously, where the fiber above b is given by the category $f(b)$. For two objects $x, y \in \mathcal{C}_f$, we declare

$$\hom_{\mathcal{C}_f}(x, y) := \{(\beta : p(x) \to p(y), \alpha \in \hom_{f(p(y))}(f(\beta)(x), y)\}.$$

Picture when B is the 2-simplex

Remark VII.2.0.3. The definition of a mapping cylinder for a continuous map $f : X \to Y$ is quite concrete: You take a disjoint union of $X \times \Delta^1$ and Y, then quotient.

You might wonder if a similar construction exists for simplicial sets. Indeed, the description of the previous paragraph applies identically when X and Y are simplicial sets. But, even when X and Y are (nerves of) categories, the resulting mapping cylinder is *not* a (nerve of a) category. For example, the edge from x to $f(x)$, followed an edge from $f(x)$ to y, does not have a 2-simplex in the cylinder exhibiting some composition. One can think of the Grothendieck construction as some universal way of turning a mapping cylinder of functors into a category.

VII.3. coCartesian edges

So we've seen how to make a "fibration" (via the Grothendieck construction) from the data of any functor $B \to \mathsf{Cat}$.

What we'd love to do is to *characterize* the functors $\mathcal{C} \to B$ that look like they arise from a Grothendieck construction. Then we'd have a hope of an equivalence of two theories: The theory of functors $B \to \mathsf{Cat}$, and the theory of certain fibrations. Mathematicians have achieved this goal

(Theorem VII.4.0.4). I want to lead you to the definition of term used for these special fibrations – the *coCartesian fibrations*.

It turns out that one can characterize such fibrations very cleanly by first understanding the case of $B = \Delta^1$. So let's do it. Fix a functor

$$\begin{tikzcd} \mathcal{C} \arrow[d, "p"] & . \\ \Delta^1 \end{tikzcd}$$

We notate by X and Y the fibers of p above $0 \in \Delta^1$ and $1 \in \Delta^1$, respectively.

The Grothendieck construction for a functor $f : X \to Y$ has the following property: For a given object x in the fiber of p above 0, there is a collection of preferred morphisms from x through which any other morphism to an object y above 1 must factor.

For example, let e be the edge from x to $f(x)$ corresponding to the morphism $\mathrm{id}_{f(x)}$. Then – if y is an object of Y – any morphism $g : x \to y$ in $\mathcal{C}_f$ must factor as $g' \circ e$ for some $g' : f(x) \to y$.

Upshot. For any object x, there is a "universal edge" e such that any function out of x with a Δ^1/horizontal direction is factored by e.

Put another way, fixing e, and given the following solid arrow h in $\mathcal{C}_f$, there always exists a dashed arrow rendering the diagram commutative.

(VII.3.1)
$$\begin{tikzcd} x_0 \arrow[r, "e"] \arrow[dr, "h"'] & y_1 \arrow[d, dashed, "\exists"] \\ & y_2 \end{tikzcd}$$

(The letter x indicates you are in the fiber above $0 \in \Delta^1$, while the letter y indicates you are in the fiber above $1 \in \Delta^1$. in the diagram which vertex of Δ^1 the objects live over.)

Remark VII.3.0.1. Such universal edges are not unique. Indeed, any edge corresponding to an automorphism of $f(x)$ would fit the bill – for example, if you fix an automorphism σ of $f(x)$, the map from x_0 to $y_1 = f(x)$ given by σ would also factor all other edges h.

That any two such universal edges are related by automorphisms is a general phenomenon. Indeed, these universal edges are related by a unique automorphism – that universal objects are only unique up to unique equivalence is a familiar phenomenon in category theory (e.g., limits and colimits).

Remark VII.3.0.2. That the choice of e is not unique is another sign of the flexibility of these fibrations: A given fibration might represent several different functors (essentially by saying which e's you prioritize), but they will all be equivalent up to natural isomorphism.

Another illustration of this phenomenon is as follows: Even if you know the functor $\Delta^1 \to \mathsf{Cat}$, the Grothendieck construction – as an isomorphism class of a category – *forgets* which of the locally p-coCartesian edges corresponds to an "identity" of an object, because it forgets which object of Y deserves to be called $f(x)$.

Remark VII.3.0.3. In fact, given e and the map $h : x_0 \to y_2$, this filler is unique; but this is because we are working with *categories*; but we will ignore this fact.

Definition VII.3.0.4. Given any functor[3] $p : \mathcal{C} \to \Delta^1$, and an edge e over Δ^1, the edge e is called a *p-coCartesian* edge over Δ^1 when the fillers (VII.3.1) always exist.

The fact that I want to suggest (and it's true) is the following: If $p : \mathcal{C} \to \Delta^1$ is a functor, and if

(C1) every object of $X = p^{-1}(0)$ is the domain of some p-coCartesian edge,

then p is equivalent to a Grothendieck construction of some functor $X \to Y = p^{-1}(1)$. For brevity, we will say that a functor $p : \mathcal{C} \to \Delta^1$ is a *coCartesian fibration* if p has enough coCartesian edges. We have the following:

Theorem VII.3.0.5 (Imprecise). There is an equivalence between (i) coCartesian fibrations over Δ^1 and maps between them that respect coCartesian edges; and (ii) Functors $\Delta^1 \to \mathsf{Cat}$ and natural transformations between them.

VII.4. coCartesian fibrations (for categories)

So, how do we think about the case of general B, and not just $B = \Delta^1$? In other words, when does a functor $p : \mathcal{C} \to B$ look like it arises from a Grothendieck construction?

The very least you'd expect of $p : \mathcal{C} \to B$ to encode a functor $B \to \mathsf{Cat}$ is that every edge of B encodes a functor – so that for every edge of B, the restriction of p to that edge admits enough p-coCartesian edges. It is common to call such a p a "locally[4] coCartesian" fibration. And any edge of $\mathcal{C}$ that is coCartesian when restricted to an edge of B will be called a *locally* p-coCartesian edge.

To see what else is necessary, let's consider the case $B = \Delta^2$ for concreteness. Suppose that you've chosen an object $x \in p^{-1}(0)$. If p is a locally coCartesian fibration, there's some locally p-coCartesian edge e_{01}, above the edge $\Delta^{\{0,1\}} \subset B$ from 0 to 1, that "realizes" the codomain y_1 of e_{01} as the image of x under some functor f_{01}. Likewise, there is a locally p-coCartesian edge e_{12} for which $p(e_{12}) = \Delta^{\{1,2\}} \subset B$ realizing some object z_2 as the image $f_{12}(y_1)$ for some functor f_{12}.

[3]This is a definition for when $\mathcal{C}_f$ and B are both categories, not ∞-categories.

[4]The "locally" here refers to the fact that the condition only refers to a single edge. When we want to define a coCartesian fibration over an arbitrary B, with potentially many edges, we will check this condition "locally" over every edge of B.

Then the *very least* we would want of p is that the composition $e_{12} \circ e_{01}$ be a locally p-coCartesian edge. Informally, we would want the composite edge to realize z_2 as the image of the composite functor $f_{12} \circ f_{01}$.

It turns out that this "very least" requirement is enough to guarantee that p encodes a functor $B \to \mathsf{Cat}$. We codify it in the following definition.

DEFINITION VII.4.0.1. Let $p : \mathcal{C} \to B$ be a functor (between ordinary categories). We say that p is a *coCartesian fibration* if the following holds:

(1) p is a locally coCartesian fibration. In other words, for every edge $\Delta^1 \to B$, the restriction of p to Δ^1 satisfies (C1).
(2) Locally coCartesian edges are closed under composition. In other words, if e and e' are locally p-coCartesian, and can be composed, then the composition $e' \circ e$ is also locally p-coCartesian.

REMARK VII.4.0.2. We have chosen – in these notes – to define a coCartesian fibration as an outcome of two checks: (i) That locally (i.e., edge-by-edge in the base) the fibration looks like it arises from a functor, and (ii) checking that this local property is closed under composition.

There is another way to characterize a coCartesian fibration using a more "global" (in B) notion of coCartesian edge. Informally, we call an edge e *p-coCartesian* if for *any simplex* $\Delta^n \to B$ for which $p(e)$ is the intial edge from 0 to 1, a partial lift of Δ^n to $\mathcal{C}$ along Λ^n_0 can be extended to a full lift of Δ^n. Then we say p is a coCartesian fibration if every object x of $\mathcal{C}$ and every edge in B out of $p(x)$ admits a p-coCartesian edge lift.

The equivalence of our "local-to-global" definition and the latter definition is Proposition 2.4.2.8 of Higher Topos Theory. (There, the equivalence is proven in the ∞-categorical setting as well.)

REMARK VII.4.0.3. A coCartesian fibration between categories is also called a *Grothendieck op-fibration*.

There is a notion of a Grothendieck fibration of categories which encodes a functor B^{op} to the category of categories; for historial reasons, those p that encode *covariant* functors from B are called op-fibrations in the literature. The notion of coCartesian fibration is a term that is also used in the ∞-categorical setting – coCartesian fibrations $p : \mathcal{C} \to B$ encode functors from an ∞-category B to the ∞-category of ∞-categories.

As advertised, the Grothendieck construction – which turns a functor $\mathcal{B} \to \mathsf{Cat}$ into a coCartesian fibration – is an equivalence. The articulation of this equivalence requires some 2-categorical language, but we are going to sweep that under the rug for the sake of exposition. (The 2-categorical structure, homotopically speaking, is quite natural – the 2-morphisms are only needed to express some natural isomorphisms, which one should think of as natural homotopy data. Indeed, the data simply helps us keep track of the fact that p-coCartesian edges are not unique, but are unique up to natural isomorphisms of objects.) Here is one articulation of the equivalence; it generalizes Theorem VII.3.0.5 to the setting where B is not a single edge.

Theorem VII.4.0.4 (Imprecise). There is an equivalence between (i) coCartesian fibrations over B and maps between them that respect coCartesian edges; and (ii) Functors $B \to \mathsf{Cat}$ and natural transformations between them.

Remark VII.4.0.5. The equivalence above, of course, tells us that we can think of a family of categories as a functor $B \to \mathsf{Cat}$, or as a coCartesian fibration over B. It turns out that the latter way of thinking is often more convenient, just because there are more constructions one can do with fibrations than with functors.

Indeed, Grothendieck invented Grothendieck fibrations (which are Cartesian, not coCartesian) to be able to think more clearly about stacks. It be came too unwiedly to think about a scheme or a stack as a functor from the site of affine schemes into sets or categories, especially when writing down what it means for a stack to be a sheaf of categories. Hence many of your friends in algebraic geometry – especially those who have thought about stacks – will be familiar with the notion of a Grothendieck fibration.

VII.5. An example, and adjunction as a property

A classic example is to consider the category $\widetilde{\mathsf{Mod}}$ whose objects are pairs (R, M) with R a commutative ring and M a module. A morphism $(R, M) \to (R', M')$ in this category can be thought of in two equivalent ways:

(i) A map of rings $f : R \to R'$, together with a map of R'-modules $R' \otimes_R M \to M'$, or
(ii) A map of rings $f : R \to R'$, together with a map of R-modules $M \to f^* M'$, where $f^* M'$ is my notation for the group M', thought of as an R-module via f.

That these two ways are equivalent is the statement that there is a free-forget adjunction, where $R' \otimes_R -$ is the "free R'-module" functor, and $f^* M$ is the "forget" functor.

Of course, there is the functor

$$\begin{tikzcd} \widetilde{\mathsf{Mod}} \arrow[d, "p"] & (R, M) \arrow[d, mapsto] \\ \mathrm{Rings} & R \end{tikzcd} .$$

It is an exercise to check that p is a coCartesian fibration; it classifies the covariant functor taking a ring to its category of modules, and sending $f : R \to R'$ to $R' \otimes_R - : R\mathsf{Mod} \to R'\mathsf{Mod}$. It turns out that p is also a Cartesian fibration, classifying the functor

$$\mathrm{Rings}^{\mathrm{op}} \to \mathsf{Cat}, \qquad R \mapsto R\mathsf{Mod}, \qquad f \mapsto f^*.$$

That p is both Cartesian and coCartesian encodes that f^* and $R' \otimes_R -$ are adjoints to each other. So already we see that adjunctions (which typically require *data* of natural transformations between two functors) can be encoded in a single *property* (of whether a map is both coCartesian and Cartesian).

Remark VII.5.0.1. This is incredibly useful. In many homotopical situations, you can imagine that it would be a pain in the but to not just have a natural transformation to exhibit a "homotopy adjunction," but to be required to provide all kinds of higher homotopies to exhibit the naturality of the adjunction.

The general theme – as illustrated by the example of adjunctions – is that we can trade "structuring data" (such as natural transformations exhibiting an adjunction) with a "property to check" (such as whether a fibration is both coCartesian and Cartesian). Such an exchange is really useful.

VII.6. Generalizing coCartesian fibrations to the setting of ∞-categories

From last time, you probably remember that to turn categorical ideas into ∞-categorical ideas, one must account for higher simplices. Here's what that looks like. Instead of a single object y_2 that we want to map to, now take $y_2, \ldots, y_k$ in the fiber above 1, and we suppose we have the following diagram (which I draw in the case $k = 2$):

(VII.6.1)
$$\begin{tikzcd}
x_0 \arrow[r, "e"] \arrow[dr] \arrow[drr] & x_1 \arrow[d] \arrow[dr] & \\
& y_1 \arrow[r] & y_2
\end{tikzcd}$$

You should visualize this as a tetrahedron with every face filled in *except* for the face spanned by X_1, Y_1, Y_2 (and without an interior of the tetrahedron). Can you fill in this tetrahedron, and in particular, exhibit that the composite map $X_1 \to Y_1 \to Y_2$ is equal to the map $X_1 \to Y_2$?

When $\mathcal{C}$ is a category, the answer is a resounding yes. But when $\mathcal{C}$ is not a category, and rather an ∞-category, many such inner horns Λ^3_0 may not be fillable. (The collection of homotopies encoded in a map from Λ^3_0 may not form, say, a contractible loop in the space of maps.)

Because we cannot keep drawing higher-dimensional simplices on paper, let us convert the filling diagram (VII.3.1) into a filling diagram one category level up; i.e., into a diagram of simplicial sets, rather than a diagram inside $\mathcal{C}$. Then, that a filler (VII.3.1) exists in $\mathcal{C}$ is equivalent to asserting a filling simplex as below exists int he category of simplicial sets:

$$
\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[dr] \arrow[drr, "e"] & & \\
& \Lambda^2_0 \arrow[r, "{e,h}"] \arrow[d] & \mathcal{C} \arrow[d] \\
& \Delta^2 \arrow[r, "s_1"] \arrow[ur, dashed, "\exists"] & \Delta^1.
\end{tikzcd}
\tag{VII.6.2}
$$

Remark VII.6.0.1. Let's first parse the solid arrows in the above diagram (VII.6.2).

- The arrow labeled e picks out the edge in $\mathcal{C}$ that we want to demonstrate is "universal;" i.e., locally p-coCartesian.
- The arrow labeled e, h picks out a horn-shapred diagram in $\mathcal{C}$, and it is exactly the (solid) diagram from (VII.3.1).
- The vertical arrows are the obvious ones: The righthand arrow is the projection $p : \mathcal{C} \to \Delta^1$ that we want to demonstrate as "arising from a Grothendieck constructoin," and the lefthand vertical arrow is the inclusion of the horn into the 2-simplex.
- The bottom horizontal arrow is to make sure that the horn Λ^2_0 lies over the correct edge. After all, there are many maps from Δ^2 to Δ^1, and we specifically choose s_1 so that the arrows $(0 \to 1)$ and $(0 \to 2)$ of Δ^2 both traverse the edge Δ^1.

  As a consequence, the bottom horizontal arrow ensures that e (and hence h) lives over the non-trivial edge of Δ^1. The diagram (VII.6.2), and especially the bottom horizontal arrow, will seem even less redundant when we replace Δ^1 by an arbitrary base B.

As motivated by the discussion before (VII.6.2), the $n = 2$ case of Λ^n_0 is all we need to check when $\mathcal{C}$ is a category. But when $\mathcal{C}$ is an ∞-category, e may not be sufficiently universal even after filling 2-horns – after all, the 3-horn depicted in (VII.6.1) may not admit a filler. And if such a 3-horn filler does not exist, we would conclude that the 2-horn fillings produced by e are not canonical (as the 2-horn fillings yield non-homotopic ways to factor the $x_0 \to y_i$ edges through e).

Here, then, is the generalization of Definition VII.3.0.4 to the setting of an ∞-category $\mathcal{C}$ mapipng to Δ^1:

Definition VII.6.0.2. Let $\mathcal{C}$ be an ∞-category. Given any functor $p : \mathcal{C} \to \Delta^1$, and an edge e over Δ^1, we say e is a *p-coCartesian* edge over Δ^1

when if for every solid diagram as below, a dashed filler always exist.

(VII.6.3)
$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[dr] \arrow[drr, "e"] & & \\
& \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
& \Delta^n \arrow[r, "s_1\circ\ldots\circ s_1"'] \arrow[ur, dashed, "\exists"] & \Delta^1.
\end{tikzcd}$$

Remark VII.6.0.3. As before, if e is a p-coCartesian edge above Δ^1, it means that any map from $x \in p^{-1}(0)$ to $y \in p^{-1}(1)$ – and in fact, from x to any homotopy coherent diagram of y_i in $p^{-1}(1)$ – will factor through e, canonically up to homotopy.

Informally, this means any diagram emanating from x factors canonically through the edge e.

Definition VII.6.0.4. More generally, for an arbitrary ∞-category $\mathcal{C}$, let $p : \mathcal{C} \to B$ be a functor where B is (the nerve of) an ordinary[5] category. We say that an edge e in $\mathcal{C}$ is a *locally p-coCartesian edge* if, after restricting p to the edge $\Delta^1 \xrightarrow{p(e)} B$, e is a p-coCartesian edge over Δ^1.

VII.7. coCartesian fibrations

As before, we would expect a map $p : \mathcal{C} \to B$ to encode a functor if (i) we have enough locally p-coCartesian edges, and (ii) such edges (which encode images of functors) compose (to encode the image of a composition of functors). We codify this as follows:

Definition VII.7.0.1 (coCartesian fibration over a category). Let $p : \mathcal{C} \to \mathcal{B}$ be a functor of ∞-categories, where $\mathcal{B} = N(B)$ is the nerve of a (ordinary) category[6]. We say that p is a *coCartesian fibration* if the following properties are satisfied:

(1) p is a locally coCartesian fibration. In other words, for every edge $\Delta^1 \to B$, the restriction of p to Δ^1 has enough coCartesian edges over Δ^1 (Definition VII.6.0.2).
(2) Locally coCartesian edges are closed under composition. In other words, if e and e' are locally p-coCartesian, and $a : \Delta^2 \to \mathcal{C}$ is any 2-simplex with $d_2 a = e, d_0 a = e'$, then $d_1 a$ is also locally p-coCartesian.

[5]When B is not an ordinary category, one must impose another condition on p – that it be an inner fibration. We will not discuss this point, but let me just say that any functor $\mathcal{C} \to B$ is an inner fibration if B is (the nerve of) an ordinary category.

[6]Again, when $\mathcal{B}$ is not an ordinary category, one simply adds on the restriction that p be an inner fibration

Remark VII.7.0.2. This is exactly the generalization of the notion of a coCartesian fibration (when $\mathcal{C}$ is the nerve of a usual category) to the setting of ∞-categories. There is also a notion of coCartesian fibration when $\mathcal{B}$ is an ∞-category (that does not arise as the nerve of an ordinary category); we will leave this for Section VII.10, as we will not need it today.

The big result that we will be using today is:

Theorem VII.7.0.3 (Vague.). A coCartesian fibration over $\mathcal{B}$ is the same thing as a functor $\mathcal{B} \to \mathcal{C}\mathrm{at}_\infty$.

Really, Lurie exhibits an equivalence between two model categories; the result is that the "same thing as" is meant to be interpreted as there is a unique, and natural, up to contractible choice, way to convert between the two kinds of data. For a detailed statement, you can see Theorem 3.2.0.1 of Higher Topos Theory.

VII.8. What is a symmetric monoidal ∞-category? Part II.

We have now seen that a functor from BLAH into $\mathcal{C}\mathrm{at}_\infty$ is the same thing as a coCartesian fibration over BLAH. So here is another attempt at defining what a symmetric monoidal ∞-category is, by taking the obvious coCartesian analogue of Definition VII.1.2.1.

Definition VII.8.0.1 (Another first attempt). A symmetric monoidal ∞-category is a coCartesian fibration

$$\mathcal{C}^{\otimes} \to \mathsf{\mathcal{F}in}$$

from some ∞-category $\mathcal{C}^{\otimes}$ to $\mathsf{\mathcal{F}in}$, such that the fiber $\mathcal{C}_I$ over an object I is identified with the product $(\mathcal{C}_{\underline{1}})^I$.

This would be an okay definition, too; but I am going to make a minor qualm: Do we really want an equality $Z(I) = Z(\underline{1})^I$ to be *built in* to our definition? In our first first attempt, such an equality seems natural, but also rather strict. What if, in some situations, it is more natural to have a category that is identified with $Z(\underline{1})^I$ up to equivalence, and not via an equality? (See Remark VII.8.2.10.)

It would be quite satisfying if something like this could be phrased not as such an identification (which is more data), but as a *property.* It turns out that the combinatorics of adjoining a basepoint to our finite sets allows us to do this.

VII.8.1. Finite pointed sets.

We follow the notation of Chapter 2 of Higher Algebra. This is to make your reading easier in case you want to cross-reference these writings with Lurie's.

Notation VII.8.1.1 ($\mathsf{\mathcal{F}in}_*$). We let

$$\mathsf{\mathcal{F}in}_*$$

denote the category of finite pointed sets.

An object of $\mathcal{F}\mathsf{in}_*$ is a finite set I, equipped with a chosen basepoint $*$ (which one might think of as a choice of function from $\underline{1} \to I$). A morphism $I \to J$ is a function that sends the basepoint to the basepoint (which one might think of as a function compatible with the functions from $\underline{1}$).

Notation VII.8.1.2. We let $\langle n \rangle := \underline{n} \coprod *$. That is,

$$\langle n \rangle = \{1, \ldots, n, *\}$$

is a set with $n+1$ elements, with one of them $*$ designated as a basepoint. Note

$$\langle 0 \rangle = \{*\}$$

is a 1-element set.

Every object of $\mathcal{F}\mathsf{in}_*$ is (non-canonically) isomorphic to an object of the form $\langle n \rangle$. So we will only consider the objects $\langle n \rangle$ for most discussions.

Example VII.8.1.3. There is a "trivial" map

$$\langle n \rangle \to \langle 0 \rangle.$$

Example VII.8.1.4 (Active maps). In the opposite extreme, any function $\underline{n} \to \underline{m}$ of finite sets induces a function

$$\langle n \rangle \to \langle m \rangle$$

by sending the basepoint to the basepoint. We call such maps *active maps.*

You should think of active maps as encoding all the structures $\mathcal{F}\mathsf{in}$ wanted to encode – e.g., multiplication, commutativity, associativity. There is another class of maps that we now identify – inert maps – that you should think of as purely formal maps that allow us to more naturally articulate the direct-product Assumption VII.1.1.3.

Definition VII.8.1.5. A function of finite pointed sets $f : \langle n \rangle \to \langle m \rangle$ is called *inert* if, for every $i \in \underline{m} = \langle m \rangle \setminus \{*\}$, the preimage $f^{-1}(i)$ has exactly one element.

Remark VII.8.1.6. You should think of an inert map as exactly the data of an injection $\underline{m} \to \underline{n}$.

Example VII.8.1.7. For every n, there are exactly n inert maps

$$\rho^i : \langle n \rangle \to \langle 1 \rangle, \qquad 1 \leq i \leq n$$

satisfying the property that $\rho^i(i) = 1$.

Example VII.8.1.8. The only inert map from $\langle 0 \rangle$ is $\mathrm{id}_{\langle 0 \rangle}$. Every map to $\langle 0 \rangle$ is inert.

Remark VII.8.1.9. I think that – in topology, at least – the category of finite pointed sets first gained attention through work of Segal. I may be wrong. (In Segal's language, the category of finite pointed sets was encoded in its opposite category, which Segal called Γ.) Segal used the combinatorics of this category to organize connective spectra. There is a very nice MathOverflow question[7] by Qiaochu Yuan, with equally nice answers, on this category.

VII.8.2. The definition of symmetric monoidal ∞-category. We will now try to fix the complaints I had about our attempted Defintion VII.8.0.1. In doing so, we arrive at Lurie's definition.

NOTATION VII.8.2.1 ($\mathcal{C}^{\otimes}_{\langle n\rangle}$). Fix an ∞-category $\mathcal{C}^{\otimes}$ and a coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$. We let

$$\mathcal{C}^{\otimes}_{\langle n\rangle}$$

denote the fiber above the object $\langle n\rangle$. More precisely, it is the simplicial subset of $\mathcal{C}^{\otimes}$ consisting of those k-simplices that are sent to the degenerate k-simplex at $\langle n\rangle$.

DEFINITION VII.8.2.2. A *symmetric monoidal ∞-category* is the data of an ∞-category $\mathcal{C}^{\otimes}$ and a coCartesian fibration

$$\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$$

satsifying the following condition:

- For each $n \geq 0$, the induced functor

$$\mathcal{C}^{\otimes}_{\langle n\rangle} \to \prod_n \mathcal{C}^{\otimes}_{\langle 1\rangle} \tag{VII.8.1}$$

  is an equivalence of ∞-categories. That is, the coCartesian fibration *exhibits* the fiber above $\langle n\rangle$ as a direct product of the fiber above $\langle 1\rangle$.

REMARK VII.8.2.3. Let's explain the "induced functor" appearing in (VII.8.1). The ability to write down this induced functor is the exact motivation for considering $\mathcal{F}\mathsf{in}_*$ instead of $\mathcal{F}\mathsf{in}$.

Fixing n, recall the inert maps ρ_i from Example VII.8.1.7. By the equivalence between coCartesian fibrations and functors to $\mathcal{C}\mathrm{at}_\infty$ (Theorem VII.7.0.3), we know that each ρ_i defines a functor

$$\mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle}.$$

In other words, by the universal property of products (as a limit of ∞-categories), these n maps define a single map

$$\mathcal{C}^{\otimes}_{\langle n\rangle} \to \prod_n \mathcal{C}^{\otimes}_{\langle 1\rangle}.$$

EXAMPLE VII.8.2.4. A sneaky case is when $n = 0$. Then there are no maps ρ^i, and the "induced map" is a map

$$\mathcal{C}^{\otimes}_{\langle 0\rangle} \to \prod_{\emptyset} \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq *.$$

[7] "Where does Segal's category come from?" `https://mathoverflow.net/questions/144328/where-does-segals-category-come-from`

(The empty direct product is a terminal object; in the ∞-category of ∞-categories, a terminal object is given by the (nerve of) the category with one object and only the identity morphism: Δ^0, here written as $*$.) Thus, Definition VII.8.2.2 demands that $\mathcal{C}^{\otimes}_{\langle 0\rangle}$ is equivalent to Δ^0 as an ∞-category.

Remark VII.8.2.5. Having the map (VII.8.1) is the main reason for the passage from $\mathcal{F}\mathsf{in}$ to $\mathcal{F}\mathsf{in}_*$. As a consequence of being able to think about $\mathcal{C}^n$ everywhere, almost every definition involving symmetric monoidal ∞-categories (and, more generally, ∞-operads) will involve not only respecting the map to $\mathcal{F}\mathsf{in}_*$, but respecting coCartesian edges that lie above the inert morphisms.

Notation VII.8.2.6. Fix a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$. We will denote the fiber $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ by $\mathcal{C}$, and call it the *underlying ∞-category.*

Notation VII.8.2.7. Let $\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$ be a symmetric monoidal ∞-category. Consider the map

$$\langle 2\rangle \to \langle 1\rangle$$

given by $1, 2, \mapsto 1$. By the coCartesian fibration assumption, there is an associated functor

$$\mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle} =: \mathcal{C}.$$

By the product-preserving assumption of (VII.8.1), we have an equivalence

$$\mathcal{C} \times \mathcal{C} \longleftarrow \mathcal{C}^{\otimes}_{\langle 2\rangle} \longrightarrow \mathcal{C}^{\otimes}_{\langle 1\rangle} =: \mathcal{C}.$$

And a choice of inverse (and homotopical data rendering the inverse an inverse) is unique up to contractible choice. Thus, we have a map

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{VII.8.2}$$

by composing the arrows in the above diagram.

Notation VII.8.2.8. Fix a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$. We will abuse notation and say that $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, suppressing the coCartesian fibration from the notation.

Remark VII.8.2.9. In traditional category theory, one would simply say "Let $\mathcal{C}$ be a symmetric monoidal category." In ∞-category literature, it is now common to say "Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category." One advantage of this notation is that it also sets the notation for the symmetric monoidal structure.

Remark VII.8.2.10 (Really?). You would be justified in being deeply troubled by the definition of the "monoidal structure" $\otimes$ in (VII.8.2). It is not very healthy to think of a symmetric monoidal structure in such terms. In practice, one of two things happen:

(I) Your symmetric monoidal ∞-category arises from an incredibly concrete model, so the maps (VII.8.1) may even be *isomorphisms* of simplicial sets. This is the case, for example, if one takes a usual symmetric monoidal category, re-writes it as a multicategory[8], then takes the nerve.

(II) You want to formally conclude the existence of a symmetric monoidal structure on an ∞-category – which means you must take $\mathcal{C}$ and construct $\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$. All formal processes boil down to checking the existence of horn-fillers of some sort (due to the definition of coCartesian fibration, for example), and you want techniques to do this. The techniques run into a wall if you demand isomorphisms of simplicial sets, and do not allow for the flexibility provided by "just check a property of the map (VII.8.1)."

As an example, a formal process that arises a lot is localization. Even in the classical setting, I would invite you to try to take a symmetric monoidal category $\mathcal{C}$, and write down an equivalence $(\mathcal{C}\times\mathcal{C})[(W\times W)^{-1}] \to \mathcal{C}[W^{-1}] \times \mathcal{C}[W^{-1}]$. The natural diagrams will involve coherences between zig-zags of squares (or squares of zig-zags). Once certain equivalence relations must be invoked, you realize that the natural structure to carry around is not that of isomorphism, but to allow for homotopical data to encode equivalences. Without modding out by equivalence relations, isomorphisms of categories are out the window.

VII.9. Some pay-offs

Phew. Wasn't that a journey? It would have been very fast to just give you the definition, but I really did want to motivate $\mathcal{F}\mathsf{in}_*$, because this category can be the most confusing part of the definition.

If someone drags you along a mathematical journey and the endpoint is a definition – but not a theorem – you can rightly ask what it was all for. Let's illustrate a nice consequence: The one big definition makes other definitions easier.

VII.9.1. Commutative algebras. Fix a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$.

Definition VII.9.1.1. A *commutative algebra* in $\mathcal{C}^{\otimes}$ is a section

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[d, "p"', shift right] \\ \mathcal{F}\mathsf{in}_* \arrow[u, bend right, shift right] \end{tikzcd}$$

that sends inert edges to p-coCartesian edges. Put another way, a commutative algebra is a horizontal arrow

$$\begin{tikzcd} \mathcal{F}\mathsf{in}_* \arrow[rr] \arrow[dr, "\mathrm{id}"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\ & \mathcal{F}\mathsf{in}_* & \end{tikzcd}$$

[8] I apologize for not covering the topic of multicategories; it is a useful language for some users of this stuff.

making the diagram of simplicial sets commute, taking inert morphisms in $\mathcal{F}\mathsf{in}_*$ to p-coCartesian edges in $\mathcal{C}^{\otimes}$.

Given such a map, we let the image of $\langle 1\rangle$ by A, and we will often call $A \in \mathcal{C}$ the commutative algebra.

Definition VII.9.1.2. A *map* of commutative algebras $A \to B$ is a functor

$$\mathcal{F}\mathsf{in}_* \times \Delta^1 \to \mathcal{C}^{\otimes}$$

respecting the projections to $\mathcal{F}\mathsf{in}_*$, and where the restrictions to $\mathcal{F}\mathsf{in}_* \times \Delta^{\{0\}}$ and $\mathcal{F}\mathsf{in}_* \times \Delta^{\{1\}}$ give rise to A and B, respectively.

Remark VII.9.1.3. Indeed, we can define the ∞-category of commutative algebras in $\mathcal{C}^{\otimes}$ now. It is the simplicial set whose k-simplices are maps

$$\mathcal{F}\mathsf{in}_* \times \Delta^k \to \mathcal{C}^{\otimes}$$

respecting projections to $\mathcal{F}\mathsf{in}_*$, and sending inert edges of each $\mathcal{F}\mathsf{in}_* \times \Delta^{\{i\}}$ to coCartesian edges.

VII.9.2. Symmetric monoidal and lax symmetric monoidal functors.

Definition VII.9.2.1. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be symmetric monoidal ∞-categories. A *symmetric monoidal functor* from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ is a horizontal arrow

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\ & \mathcal{F}\mathsf{in}_* & \end{tikzcd}$$

taking p-coCartesian edges to q-coCartesian edges, and making the indicated triangle of simplicial sets commute.

The definition of algebra and of symmetric monoidal functor differ only in which coCartesian edges are respected. Indeed, relaxing the symmetric monoidal functor condition, we arrive at the following:

Definition VII.9.2.2. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be symmetric monoidal ∞-categories. A *lax symmetric monoidal functor* from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ is a horizontal arrow

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\ & \mathcal{F}\mathsf{in}_* & \end{tikzcd}$$

taking p-coCartesian edges of inert morphisms to q-coCartesian edges (of inert morphisms), and making the indicated triangle of simplicial sets commute.

VII.10. (Not covered in spoken lecture) coCartesian fibrations (over ∞-categories)

Definition VII.10.0.1 (coCartesian fibration over an ∞-category). Let $p : \mathcal{C} \to \mathcal{B}$ be a functor of ∞-categories.[9] We say that p is a *coCartesian fibration* if the following properties are satisfied:

(a) For every edge $f : b \to b'$ in $\mathcal{B}$, and every object x with $p(x) = b$, there exists a p-coCartesian edge e for which $p(e) = f$ and $d_1 e = x$.

(b) For every $n \geq 2$ and $0 < k < n$, and for every solid diagram as below, a dashed lift exists:

(VII.10.1)
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

(The term for this is that p is an "inner fibration.")

Remark VII.10.0.2. The concept of an inner fibration has no classical category theory analogue that's easy to state. But let me just note two things about inner fibrations. First, if $p : \mathcal{C} \to \mathcal{B}$ is an inner fibration, consider the solid diagrams in (VII.10.1) for which the map $\Delta^n \to \mathcal{B}$ is constant. Then the fibers of p are all ∞-categories.

Second, it turns out that if p is an inner fibration, then for every $a : \Delta^1 \to \mathcal{B}$, you can interpret $p^{-1}(a)$ as encoding a *bimodule*[10] between its two fiber ∞-categories; that is, something like a functor $p^{-1}(0)^{\mathrm{op}} \times p^{-1}(1) \to \mathsf{Spaces}$.

Thus, if you are already familiar with the idea that some bimodules are special and deserve to be called graphs, or functors, you can imagine that the "coCartesian edges exist" condition to be the condition that ensures p does not encode some complicated system of bimodules, but just an honest functor $\mathcal{B} \to \mathcal{C}\mathrm{at}_\infty$.

[9] Note Suppose $\mathcal{B}$ is an arbitrary ∞-category.

[10] Otherwise known as a correspondence.

Exercises

VII.11. Adjunctions

Let $\mathcal{B}$ be a category and suppose you have a functor $f : \mathcal{B}^{\mathrm{op}} \to \mathsf{Cat}$ in the category of categories. (Note the contravariance.)

(a) Construct a category $\mathcal{C}_f$ and a functor

$$\begin{tikzcd} \mathcal{C}_f \arrow[d, "p"] \\ \mathcal{B} \end{tikzcd}$$

for which p is a *Cartesian* fibration.

REMARK VII.11.0.1. This means that for every edge $q : b_0 \to b_1$ in $\mathcal{B}$, and every object $\widetilde{b}_0 \in \mathcal{C}_f$ with $p(\widetilde{b}_0) = b_0$, there exists an edge e with domain $\widetilde{b_0}$ and with $p(e) = q$, for which the following horn-filler can always be found:

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[dr] \arrow[drr, "e"] & & \\ & \Lambda^n_n \arrow[r] \arrow[d] & N(\mathcal{C}_f) \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & N(\mathcal{B}). \end{tikzcd}$$

(That is, we can get the dashed arrow whenever we are given the solid arrows.)

(b) Suppose you have a category $\mathcal{C}$ and a functor $p : \mathcal{C} \to \Delta^1$ which is both a Cartesian and coCartesian fibration. Convince yourself that p encodes an adjunction going between two categories $p^{-1}(0)$ and $p^{-1}(1)$.

(c) Now suppose you have an ∞-category equipped with a functor $p : \mathcal{C} \to \Delta^1$ which is both a Cartesian and coCartesian fibration. Convince yourself that p seems to encode what one might mean by a homotopical version of an adjunction.

VII.12. Automatic inner fibrations

Let $p : \mathcal{C} \to \mathcal{B}$ be a functor, and suppose that $\mathcal{B}$ is the nerve of an ordinary category. Then show that p is automatically an inner fibration – see (VII.10.1).

VII.13. Being locally coCartesian is not coCartesian

Find an example of a functor between ordinary categories $p : \mathcal{C} \to \Delta^2$ such that p is locally coCartesian (meaning the restriction of p to any edge of Δ^2 is coCartesian) but for which p itself is not coCartesian. (Hint: What would be associated to a composition of two functors, equipped with a natural transformation to a third functor?)

VII.14. Associativity and commutativity

Fix an ∞-category $\mathcal{C}$. Write out the diagrams in $\mathcal{C}$ that emerge from a functor $\mathcal{F}\mathsf{in} \to \mathcal{C}$, under Assumption VII.1.1.3, and for the diagrams involving only the objects $\underline{3}, \underline{2}, \underline{1}$.

VII.15. Stratified tangent bundle structures

In his talks, Mohammed talked about a homotopical way to think about the manifolds obtained from compactifying moduli spaces in Floer theory.

Here's one framework for it. At every step of this exercise is: Make sure you understand each statement.

(a) Note that there are natural inclusions of spaces $BO(0) \to BO(1) \to BO(2) \to BO(3) \to \ldots$, essentially by direct sum – a k-plane in $\mathbb{R}^\infty$ defines a $(k+1)$-plane in $\mathbb{R}^{\infty+1} \cong \mathbb{R}^\infty$. This defines a coCartesian fibration
$$\widetilde{BO} \to \mathbb{Z}_{\geq 0}.$$
where the fiber above k is the Kan complex modeling $BO(k)$. Here, $\mathbb{Z}_{geq0}$ is the poset of non-negative integers, considered as an ∞-category by taking its nerve.

(b) Fix a topological manifold with corners – more precisely, a space X equipped with a stratification $X \to \mathbb{Z}_{\geq 0}$, which means a continuous map where the codomain is given the Alexandroff topology of the poset. We demand that X have the property that it have local Euclidean charts exhibiting a topological-manifold-with-corners structure respecting this stratification.

(c) Moreover, given such a data, there is a notion of *exit path ∞-category* of X, which I will denote by $\mathrm{Exit}(X)$. This is a simplicial set sitting inside the Kan complex $\mathsf{Sing}(X)$, and there exists a natural functor $\mathrm{Exit}(X) \to \mathbb{Z}_{\geq 0}$. (Such a functor respecting stratification would not exist from $\mathsf{Sing}(X)$.)

(d) If we know that each stratum of X is smooth a priori (this is the case in Floer theory) then each stratum X_i[11] comes equipped with a map $X_i \to BO(i)$ classifying the tangent bundle.

(e) One could imagine, then, the choice of a map T

$$\begin{tikzcd} \mathrm{Exit}(X) \arrow[rr, "T"] \arrow[dr] & & \widetilde{BO} \arrow[dl] \\ & \mathbb{Z}_{\geq 0} & \end{tikzcd}$$

could be a homotopy-theoretic stand-in that is (far weaker than, but gives intution for) a smooth structure on X itself.

[11]This is the fiber above $i \in \mathbb{Z}_{\geq 0}$, otherwise known as the dimension i part of X

Exercises on symmetric monoidal ∞-categories

VII.16. $\mathcal{F}\mathsf{in}_*$

Consider the identity functor $\mathcal{F}\mathsf{in}_* \to \mathcal{F}\mathsf{in}_*$. Does this exhibit $\mathcal{F}\mathsf{in}_*$ as a symmetric monoidal ∞-category? If so, what name do you want to give it?

VII.17. Operads

If you get confused by this exercise, or need some prerequisites, read the introduction to Chapter 2 of Higher Algebra.

(a) Given a symmetric monoidal category $\mathcal{C}$ in the usual sense, write out $\mathcal{C}$ as a multicategory in the usual way. (This assumes some prior knowledge.)
(b) Show that the nerve of $\mathcal{C}$ admits a natural map to $\mathcal{F}\mathsf{in}_*$.
(c) Convince yourself that any symmetric monoidal category gives rise to an symmetric monoidal ∞-category via the above construction.
(d) Consider a symmetric monoidal functor $\mathcal{C} \to \mathcal{D}$. Can you write down the associated map of symmetric monoidal ∞-categories? How does this map play with the coCartesian edges?

VII.18. Let's reLax

Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ be symmetric monoidal ∞-category. A lax symmetric monoidal functor f will induce, for every $X, Y \in \mathcal{C}$, an arrow between $f(X \otimes Y)$ and $f(X) \otimes f(Y)$. In which direction does the arrow go?

VII.19. Commutative algebras

(a) Play around with Definition VII.9.1.1. Convince yourself that the definition indeed captures (for you) the intuition of a commutative algebra that is commutative up to homotopy.
(b) What happens if you demand that the map $\mathcal{F}\mathsf{in}_* \to \mathcal{C}^{\otimes}$ send all edges (not just the inert edges) to coCartesian morphisms?
(c) Straight from the definitions, show that a symmetric monoidal functor $f : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ preserves commutative algebras. (That is, for any commutative algebra A in $\mathcal{C}^{\otimes}$, $f(A)$ is naturally a commutative algebra in $\mathcal{D}^{\otimes}$.)
(d) Straight from the definitions, show that a lax symmetric monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ preserves commutative algebras.

(e) In the definition of maps of commutative algebras (Definition VII.9.1.2), what happens if you demand that for all $n \geq 0$, the edges of the form $\mathrm{id}_{\langle n \rangle} \times \Delta^1$ are all coCartesian?

(f) More generally, given an ∞-operad $\mathcal{O}$, an $\mathcal{O}$-algebra in $\mathcal{C}^{\otimes}$ is the data of a functor $\mathcal{O} \to \mathcal{C}^{\otimes}$ that respects the projection to $\mathcal{F}\mathsf{in}_*$, and preserves the inert coCartesian edges. Show that a lax symmetric monoidal functor preserves $\mathcal{O}$-algebras.

VII.20. $\mathbb{E}_\infty$

Consider the topologically enriched multicategory associated to the operad $\mathbb{E}_\infty$. By taking its homotopy coherent nerve, exhibit a functor to $\mathcal{F}\mathsf{in}_*$. Show that this is an equivalence of ∞-categories.

In particular, show that an $\mathbb{E}_\infty$-algebra in a symmetric monoidal ∞-category is the same thing as a commutative algebra. More precisely, exhibit an equivalence of ∞-categories between these two kinds of algebras.

LECTURE VIII

(Bonus lecture) ∞-operads

We didn't have time in our seven lectures to talk about ∞-operads. So I'll include this as a bonus chapter in the notes.

Why didn't we have time? Well, we went nice and slow so that the central ideas of our lectures came through, and so that everything was motivated coherently. And to do justice to ∞-operads, we have to introduce one more classical idea: The theory of colored operads, also known as multicategories. (These are synonyms.)

REMARK VIII.0.0.1. It may be helpful to think of the following analogy: Just as an algebra is a category with one object, an operad is a multicategory (or colored operad) with one color.

VIII.1. Colored operads, a.k.a. multicategories

Operads are good for encoding algebraic structures with one output. (So, for example, a single operad could not encode bialgebras, which have both algebra and coalgebra – two output – structures.) But even one-output operations can have interesting additional data we want to contemplate. The most common example of this is the theory of modules: It's one thing to give an algebra A, but is there an operad-like structure whose algebras encode both algebras and modules over them?

EXAMPLE VIII.1.0.1. This desire to encode modules becomes even more interesting when we recognize that different kinds of algebras can have different kinds of modules.

(a) Consider for a moment the open upper-half-plane $\{(x_1, \dots, x_n), x_n > 0\}$. Consider further an assignment which, to every open disk in $\mathbb{R}^n$ that intersects the upper-half-plane and does not intersect the origin, a copy of a chain complex A, while to any such open disk that intersects the origin, one assigns another chain complex M. Then the pair (A, M) would represent a module M with an action from an $\mathbb{E}_n$-algebra A.
(b) Or, one could imagine that A lives on all of $\mathbb{R}^n$ (not just the upper-half-plane) and consider a structure that assigns M to any open disk containing the origin. Then the pair (A, M) encodes a module M over A, but there is a sphere's worth of compatible module actions on M (the sphere being the copy of $S^{n-1} \subset \mathbb{R}^n$ about the origin). This is a different kind of module action than that of the previous paragraph.

(c) Yet another variant is to assign M to any disk whose center is contained in the hyperplane $x_n = 0$. Then (A, M) is a pair which also assigns to M an $\mathbb{E}_{n-1}$-algebra structure (on top of a module action of A on M).

Pictures would be great here

As the example above shows, it is sometimes useful to have multiple "kinds" of objects encoded. (In the above example, A represented one kind of object – an $\mathbb{E}_n$-algebra – and M another – a module of some flavor.) And different kinds of objects may admit interesting geometric or algebraic operations. This leads to the creation of the notion of a (symmetric) *colored operad*, which some people refer to as a (symmetric) *multicategory*.

I will only give an informal definition here. A succinct definition is given in Definition 2.1.1.1 of Lurie's Higher Algebra. Another source is Tom Leinster's book[1], though the perspective there is non-symmetric by default.

Definition VIII.1.0.2 (Informal). A (symmetric) *multicategory* $\mathcal{O}$ is the data of

(1) A collection $\operatorname{Ob}\mathcal{O}$ which we call the set of *colors*, or the set of *objects*.
(2) For every finite set I, every collection $\{X_i\}_{i\in I}$ of colors, and every color Y, a set of *multimorphisms*
$$\operatorname{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y),$$
(3) Finally, for every function $\alpha : I \to J$, for every collection $\{X_i\}_{i\in I}$, $\{Y_j\}_{j\in J}$, and for every choice of object Z, a composition map
$$\left(\prod_{j\in J}\operatorname{Mul}_{\mathcal{O}}(\{X_i\}_{i\in\alpha^{-1}(j)}, Y_j)\right)\times \operatorname{Mul}_{\mathcal{O}}(\{Y_j\}_{j\in J}, Z) \to \operatorname{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Z).$$

These data must satisfy various compatibilities with respect to composition of finite-set functions $I \to J \to K$ – in particular, one sees that automorphisms of a given set I acts on $\operatorname{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$, and that the composition map above is associative in an appropriate sense.

Finally, for a unital version, one requires distinguished elements in the $\#I = 1$ multimorphism sets $\operatorname{Mul}_{\mathcal{O}}(\{X\}, X)$ satisfying natural unit conditions with respect to the composition maps.

Definition VIII.1.0.3. A map $f : \mathcal{O} \to \mathcal{P}$ of multicategories is the data of

- A function $f : \operatorname{Ob}\mathcal{O} \to \operatorname{Ob}\mathcal{P}$,
- For every finite set I, for every collection $\{X_i\}_{i\in I}$ in $\operatorname{Ob}\mathcal{O}$, and for every object $Y \in \operatorname{Ob}\mathcal{O}$, a function
$$\operatorname{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y) \to \operatorname{Mul}_{\mathcal{P}}(\{f(X_i)\}_{i\in I}, f(Y))$$

[1]Tom Leinster, *Higher Operads, Higher Categories*, London Mathematical Society Lecture Note Series 298, Cambridge University Press (2004), ISBN 0-521-53215-9. Also available at arXiv:math/0305049.

respecting the composition operations of $\mathcal{O}$ and $\mathcal{P}$. (And, for f to be a unital map, units must be respected.)

Example VIII.1.0.4 (Symmetric monoidal categories). Let $\mathcal{C}$ be a symmetric monoidal category. Then there is an associated multicategory whose set of objects is the same as that of $\mathcal{C}$, and where we define

$$\mathrm{Mul}(\{X_i\}_{i\in I}, Y) := \hom_{\mathcal{C}}(\bigotimes_{i\in I} X_i, Y).$$

Example VIII.1.0.5 (Categories). Let $\mathcal{C}$ be a category. Then there is an associated multicategory whose set of objects is the same as that of $\mathcal{C}$, and where we define

$$\mathrm{Mul}(\{X_i\}_{i\in I}, Y) := \begin{cases} \hom_{\mathcal{C}}(X_i, Y) & \#I = 1 \\ \emptyset & \#I \neq 1. \end{cases}$$

Example VIII.1.0.6 (Operads). Let $\mathcal{O}$ be an operad. Then there is an associated multicategory whose set of objects $\mathrm{Ob}\,\mathcal{O} = \{*\}$ has exactly one element – so we may associate the data of a collection $\{X_i\}_{i\in I}$ simply with I itself – and where we declare

$$\mathrm{Mul}(I, *) := \mathcal{O}(\#I).$$

Example VIII.1.0.7. Let $\mathcal{C}'$ be the multicategory associated to a symmetric monoidal category $\mathcal{C}$, and $\mathcal{O}'$ the multicategory associated to an operad $\mathcal{O}$. Then a unital map of multicategories $\mathcal{O}' \to \mathcal{C}'$ is a choice of $\mathcal{O}$-algebra in $\mathcal{C}$.

Remark VIII.1.0.8. There is an obvious notion of multi-category one obtains after enriching; that is, given a symmetric monoidal category $\mathcal{V}^{\otimes}$, one can define a multi-category $\mathcal{O}$ enriched in $\mathcal{V}^{\otimes}$ by declaring each $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ to be an object in $\mathcal{V}$, and by replacing every direct product appearing in the composition map by the symmetric monoidal product $\otimes$.

One common case of interest is to take $\mathcal{V}^{\otimes}$ to be the category of spaces with direct product, or the category of chain complexes with tensor product.

Remark VIII.1.0.9. The enriched versions of ∞-operads, however, are a bit less obvious to set up. See the work of Gepner-Haugseng [2].

Example VIII.1.0.10. We will now construct a multicategory enriched in spaces that classifies the algebraic data of an $\mathbb{E}_n$-algebra acting "from every direction" on a single module. This is Example VIII.1.0.1(b).

Let $\mathcal{O}$ be a multi-category with two objects/colors, which I will call $\mathbb{A}$ and $\mathbb{M}$. Then any collection of objects $\{X_i\}_{i\in I}$ may be rewritten as $\{X_{i'} = \mathbb{M}\}_{i\in I'}$ and $\{X_{i''} = A\}_{i''\in I''}$ for $I = I' \coprod I''$. I then declare

$$\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$$

to be as follows:

[2]

- When the target color Y is equal to $\mathbb{A}$,
  - Empty when $I' \neq \emptyset$
  - The collection of rectilinear embeddings of $\coprod_{I''}[0,1]^n$ into $[0,1]^n$.
- When the target color Y is equal to $\mathbb{M}$,
  - The collection of rectilinear embeddings j of $\coprod_I [0,1]^n$ into $[0,1]^n$ for which
    (i) $i \in I'$ implies that j takes the center of the ith cube to the center of the codomain. (In particular, the multimorphism space is empty if $\#I' \geq 2$.)
    (ii) $i \in I''$ implies that the image of j restricted to the ith cube is disjoint from the center of the codomain.

VIII.2. E pluribus unum

You probably looked at the definition of multicategory and thought: "Why is this even necessary? Can I not just encode this in a single category?"

For concreteness, fix a multicategory $\mathcal{C}$. Then one could dream of a category $\mathcal{C}^{\otimes}$ whose set of objects is not $\operatorname{Ob}\mathcal{C}$, but whose set of objects is the collection of all pairs $(I, \{X_i\}_{i\in I})$, and where

$$\hom_{\mathcal{C}^{\otimes}}(\{X_i\}_{i\in I}, \{Y\}) = \operatorname{Mul}_{\mathcal{C}}(\{X_i\}_{i\in I}, Y).$$

Then, very quickly, one runs into the following question: What happens when the codomain consists of a non-singleton collection – e.g., $\{Y_j\}_{j\in J}$ for $\#J \neq 1$? What should homs into such a thing be?

If we want to reproduce a multicategory $\mathcal{C}$ from our putative category $\mathcal{C}^{\otimes}$, we'd like to encode a way in which there is no new information in such morphism spaces. Given that we are indexing our maps by functions between finite sets anyway, here is a natural way to do so:

We *declare* that a morphism from $\{X_i\}_{i\in I}$ to $\{Y_j\}_{j\in J}$ is a pair

$$(\alpha : I \to J\,,\, \{f_j\}_{j\in J})$$

where α is a function, and f_j is an element of the multimorphism space determined by α and j:

$$f_j \in \operatorname{Mul}_{\mathcal{C}}(\{X_i\}_{i\in\alpha^{-1}(j)}, Y_j).$$

Remark VIII.2.0.1. Pictorically, one might imagine that an arbitrary map from $\{X_i\}_{i\in I}$ to $\{Y_j\}_{j\in J}$ can be represented by a directed graph (and not a directed tree) having I as input set and J as output set. The data of α above instead allows us to visualize the map as draw-able in a very specific kind of (possibly disconnected) directed graph: As a forest. That is, for every $j \in J$, one draws a tree with j as a root, and with $\alpha^{-1}(j)$ as the set of leaves. The most faithful drawing for us – following Remark III.7.1.1 – would be to draw these trees as corollas.

What we arrive at is, then, is a natural way to encode a multicategory $\mathcal{C}$ into a single category $\mathcal{C}^{\otimes}$:

Construction VIII.2.0.2. Given a multicategory $\mathcal{C}$, the category associated to $\mathcal{C}$ is notated $\mathcal{C}^{\otimes}$, and defined as follows:

- The objects of $\mathcal{C}^{\otimes}$ are pairs $(I, I \to \mathrm{Ob}\,\mathcal{C})$. (That is, a finite set I together with a collection of colors $\{X_i\}_{i\in I}$.
- A morphism from $(I, I \to \mathrm{Ob}\,\mathcal{C})$ to $(J, J \to \mathrm{Ob}\,\mathcal{C})$ is a pair
$$(\alpha : I \to J, \{f_j \in \mathrm{Mul}_{\mathcal{C}}(\{X_i\}_{i\in\alpha^{-1}(j)}, Y)\}_{j\in J}$$
.

This associated category has a forgetful functor to $\mathcal{F}\mathsf{in}$.

Now, the question is – can we characterize those functors $\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}$ that realizes $\mathcal{C}^{\otimes}$ as equivalent to something arising from a multicategory? As in the previous lecture, it will be very useful to consider not $\mathcal{F}\mathsf{in}$ but $\mathcal{F}\mathsf{in}_*$. Moreover, it'll be useful to pick out the classes of morphisms in $\mathcal{F}\mathsf{in}_*$ that introduce "no new data" on our morphisms.

VIII.3. Inert maps

You already encountered inert maps in Definition VII.8.1.5. Let me recall the definition here:

Definition VIII.3.0.1. Let I_+ and J_+ be two finite pointed sets. We will say a map $\alpha : I_+ \to J_+$ is *inert* if for every $j \in J$, $\alpha^{-1}(j)$ consists of exactly one element.

Remark VIII.3.0.2. One can think of an inert map as defining an injection from J to I. Alternatively, an inert map is a quotient map $I_+ \to J_+$ obtained by identifying some elements of the domain with the base point.

Notation VIII.3.0.3. We let $\mathcal{F}\mathsf{in}_*^{\mathrm{inert}} \subset \mathcal{F}\mathsf{in}_*$ denote the subcategory consisting of the same objects, but only of the inert morphisms.

In defining symmetric monoidal ∞-categories $p : \mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$, we picked out some of the inert morphisms – namely, the morphisms $\rho_i : I_+ \to \langle 1\rangle$ – to be able to characterize the fibers $\mathcal{C}_{I_+}$ as I-fold products of $\mathcal{C}^{\otimes}_{\langle 1\rangle}$. Importantly, we already assumed the map $\mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$ was a coCartesian fibration. But not every multicategory arises from a symmetric monoidal category, so we will relax this assumption so that p is only coCartesian when restricted to the inert subcategory.

VIII.3.1. Characterizing the maps that arise from multicategories.

Proposition VIII.3.1.1. Suppose $\mathcal{C}^{\otimes}$ is a category equipped with a functor $p : \mathcal{C}^{\otimes} \to \mathcal{F}\mathsf{in}_*$. Then $\mathcal{C}^{\otimes}$ is the category associated to a multicategory $\mathcal{C}$ with category of colors $p^{-1}(\underline{1})$ if and only if the following hold:

(a) The maps $\rho_j : J_+ \to \langle 1\rangle$ (sending j to 1 and everything else to the base point) admit enough coCartesian lifts. In particular, each ρ_j defines a functor $p^{-1}(J_+) \to \rho^{-1}(\langle 1\rangle)$.

(b) The induced functor

$$\rho^{-1}(J_+) \to \prod_{j \in J} \mathcal{C}^{\otimes}_{\langle 1 \rangle}$$

is an equivalence of categories. (That is, realizes $\rho^{-1}(J_+)$ as the J-fold product of $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$. We think of $\mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ as the category of colors and 1-input operations of a multicategory.)

(c) For any morphism of finite sets $\alpha : I \to J$ and any objects $X_I \in p^{-1}(I_+)$, $Y_J \in p^{-1}(J_+)$, let $\hom^{\alpha}_{\mathcal{C}^{\otimes}}(X_I, J_Y)$ denote the collection of moprhisms from X_I to Y_J that map to α in $\mathcal{F}\mathsf{in}_*$. Then the map

$$\hom_{\mathcal{C}^{\otimes}}(X_I, Y_J) \to \prod_{j \in J} \hom_{\mathcal{C}}(X_{\alpha^{-1}(j)}, Y_j)$$

is a bijection.

What the proposition tells us is that we can indeed encode the data of a multicategory (i.e., colored operad) as the data of a single category, equipped with a map to $\mathcal{F}\mathsf{in}_*$, satisfying some properties.

VIII.4. Definition of ∞-operad

VIII.5. Trees versus finite sets

Exercises

VIII.6. Multicategories and algebras over operads

Verify Example VIII.1.0.7.

VIII.7. $\mathbb{E}_n$ acting on $\mathbb{E}_k$

For each example in Example VIII.1.0.1, construct a multicategory for which maps out of the multicategory classify algebras and modules of the given flavor.

(We tackled one such multicategory in Example VIII.1.0.10.)