

# Lecture 36

## Taylor Polynomials

### 36.1 Derivatives of polynomials are easy to compute at certain places

**Exercise 36.1.1.** Consider the function

$$T(x) = 5 + (x - \pi) + 9(x - \pi)^2 + 17(x - \pi)^3 + 19(x - \pi)^4.$$

This is a degree four polynomial.

- (a) Compute  $T(\pi)$ . **Hint:** Do *not* expand out the powers of  $x - \pi$ .
- (b) Compute  $T'(\pi)$ . **Hint:** Do *not* expand out the powers of  $x - \pi$ .
- (c) Compute  $T''(\pi)$ . **Hint:** ... Guess the hint.
- (d) Compute the *third* derivative of  $T$  at  $x = \pi$ . (The third derivative is the derivative of the second derivative.) This is sometimes written  $T'''(\pi)$ , or sometimes  $T^{(3)}(\pi)$ .
- (e) Compute  $T^{(4)}(\pi)$ . That is, compute the fourth derivative of  $T$  at  $x = \pi$ .

**Exercise 36.1.2.** Suppose you have a degree four polynomial of the form

$$T(x) = b_0 + b_1(x - a) + \frac{b_2}{2}(x - a)^2 + \frac{b_3}{3!}(x - a)^3 + \frac{b_4}{4!}(x - a)^4$$

where  $a, b_0, b_1, b_2, b_3, b_4$  are real numbers.

(Some notes:

- Remember that  $4!$  is a “factorial.” It is a shorthand for the expression  $4 \times 3 \times 2 \times 1$ . Likewise,  $3! = 3 \times 2 \times 1$ .
- If it helps, you can choose to replace  $b_0, b_1$ , and so forth with concrete numbers like 12 or  $\pi$ . But I want you to get practice reasoning without making those kinds of substitutions. The important point here is that  $a, b_0, b_1, b_2, b_3, b_4$  are *not* numbers that change with  $x$ ; they are constants.

End of notes.)

(a) Compute  $T(a)$ .

(b) Compute  $T'(a)$ .

(c) Compute  $T''(a)$ .

(d) Compute  $T^{(3)}(a)$ .

(e) Compute  $T^{(4)}(a)$ .

**Exercise 36.1.3.** Suppose somebody tells you they have a function  $f(x)$ , and that

- $f(0) = 1$ .
- $f'(0) = 0$ .
- $f''(0) = -1$ .
- $f^{(3)}(0) = 0$ .
- $f^{(4)}(0) = 1$ .

(a) Can you find a degree four polynomial  $T(x)$  such that

- $f(0) = T(0)$ ,
- $f'(0) = T'(0)$ ,
- $f''(0) = T''(0)$ ,
- $f^{(3)}(0) = T^{(3)}(0)$ , and
- $f^{(4)}(0) = T^{(4)}(0)$ ?

(b) Would you expect the graphs of  $T(x)$  and  $f(x)$  to be related in any way? Why or why not?

### 36.1.1 The definition

**Definition 36.1.4.** Let  $f$  be a function, and choose a real number  $a$ . The  $n$ th degree Taylor polynomial of  $f$  at  $a$  is the degree  $n$  polynomial  $T_n$  satisfying

- $T(a) = f(a)$ ,
- $T'(a) = f'(a)$ ,
- $\dots$ ,
- $T^{(n)}(a) = f^{(n)}(a)$ .

In other words,  $T_n$  is the polynomial whose value, derivative, second derivative,  $\dots$ , and  $n$ th derivative at  $a$  all agree with those of  $f$  at  $a$ .

Based on the previous page, we know that the  $n$ th degree Taylor polynomial can be written as

$$T_n(x) = b_0 + b_1(x - a) + \frac{b_2}{2}(x - a)^2 + \dots + \frac{b_n}{n!}(x - a)^n$$

where  $b_1 = f'(a)$ ,  $b_2 = f''(a)$ ,  $\dots$ ,  $b_n = f^{(n)}(a)$ . For example, the coefficient in front of  $(x - a)^4$  is given by  $\frac{f^{(4)}(a)}{4!}$ .

And, on the previous page, you were finding the 4th degree Taylor polynomial to some mystery function  $f$ .

**Exercise 36.1.5** (If there is time). Can you think of a trig function  $f$  that satisfies the conditions from the previous page? That is, so that

- $f(0) = 1$ .
- $f'(0) = 0$ .
- $f''(0) = -1$ .
- $f^{(3)}(0) = 0$ .
- $f^{(4)}(0) = 1$ .

## 36.2 Example: cosine

Let  $f(x) = \cos(x)$ . We'll find the Taylor polynomials of  $f$  at  $a = 0$ . We'll also plot the graph of  $T_n$  next to the graph of  $\cos(x)$  to compare.

### 36.2.1 Degrees 0 and 1

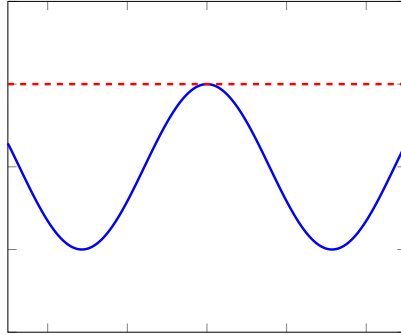
Because  $f(0) = 1$  and  $f'(0) = 0$ , we can find the degree 0 and degree 1 Taylor polynomials as follows:

$$\begin{aligned}T_0(x) &= 1 \\T_1(x) &= 1 + \cos'(0)(x - 0) \\&= 1 + 0(x - 0) \\&= 1\end{aligned}$$

(Note that even though  $T_1$  is called “degree one,” it doesn't have a linear term, because the coefficient in front of  $(x - a)$  turns out to be zero.)

Here is the graph of  $T_0$  and  $T_1$  (in dashed red) along with the graph of  $\cos(x)$  (in

solid blue):



### 36.2.2 Degree 2 (and 3)

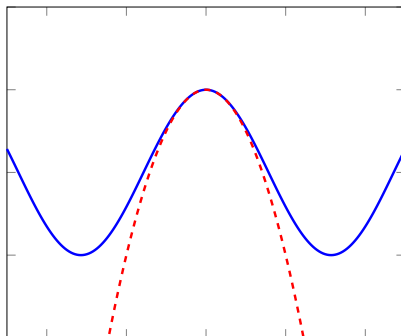
Because  $f''(0) = -1$  and  $f^{(3)}(0) = 0$ , we can find the degree 2 and degree three Taylor polynomials as follows:

$$\begin{aligned}
 T_2(x) &= 1 + 0(x - 0) + \frac{\cos''(0)}{2}(x - 0)^2 \\
 &= 1 + \frac{-1}{2}x^2 \\
 T_3(x) &= 1 + \frac{-1}{2}x^2 + \frac{\cos'''(0)}{6}(x - 0)^3 \\
 &= 1 + \frac{-1}{2}x^2 + \frac{0}{6}(x - 0)^3 \\
 &= 1 + \frac{-1}{2}x^2
 \end{aligned}$$

(Note that even though  $T_3$  is called “degree three,” it doesn’t have a degree three term, because the coefficient in front of  $(x - a)^3$  turns out to be zero.)

Here is the graph of  $T_2(x)$  (which happens to be the same as  $T_3(x)$  in this example)

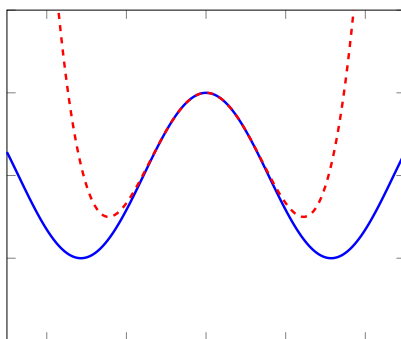
along with the graph of cosine:



### 36.2.3 Degree 4 (and 5)

$$\begin{aligned}
 T_4(x) &= 1 + \frac{-1}{2}x^2 + \frac{\cos^{(4)}(0)}{24}(x-0)^4 \\
 &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 \\
 T_5(x) &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 + 0x^5 \\
 &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4
 \end{aligned}$$

Here is the graph of  $T_4$  next to the graph of cosine:

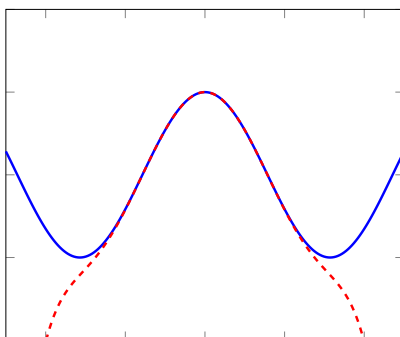


Are the graphs starting to look similar?

### 36.2.4 Degree 6

$$\begin{aligned} T_6(x) &= 1 + 0x + \frac{-1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 + 0x^5 - \frac{1}{720}x^6 \\ &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 + \frac{-1}{720}x^6 \end{aligned}$$

Here is the graph of  $T_6$  next to the graph of cosine:



**The take-away:** Taylor polynomials allow you to *approximate* a complicated function  $f$  by a simpler function (a polynomial), just by knowing the higher derivatives of  $f$  at a point  $a$ . As you can see from the graphs above, these approximations do a very good job *near*  $a$ . Further away from  $a$ , the polynomials may behave very differently from  $f$ .

## 36.3 Application: Approximating $\cos(0.5)$

Most of us do not know what  $\cos(0.5)$  is off the top of our heads. But we saw in our previous drawings that the Taylor polynomials have graphs that are very similar to the graph of  $\cos(x)$  when we are near  $a = 0$ .

So what if we evaluate  $T_n(0)$ ? We get the following numbers:

- $T_0(0.5) = 1$
- $T_2(0.5) = 1 + \frac{-1}{2}(0.5)^2 = 0.875$
- $T_4(0.5) = 0.87760416667$
- $T_6(0.5) = 1 + \frac{-1}{2}(0.5)^2 + \frac{1}{24}(0.5)^4 + \frac{-1}{720}(0.5)^6 = 0.87758246528$

- $T_8(0.5) = 1 + \frac{-1}{2}(0.5)^2 + \frac{1}{24}(0.5)^4 + \frac{-1}{720}(0.5)^6 + \frac{1}{40320}(0.5)^8 = 0.87758256216$

Try comparing these to what your calculator says  $\cos(0.5)$  is. I think you'll be pleased!

## 36.4 Preparation for next time

For next time, I expect you to be able to do the following.

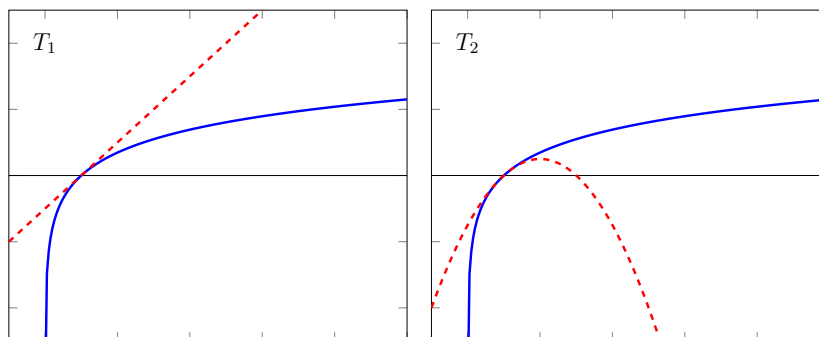
Let  $f(x) = \ln x$ .

- Compute  $f(1)$ .
- Compute  $f'(1)$ .
- Compute  $f''(1)$ .
- Compute  $f^{(3)}(1)$ .
- Compute  $f^{(4)}(1)$ .
- Write the fourth degree Taylor polynomial  $T_4(x)$  of  $\ln x$  at  $a = 1$ .

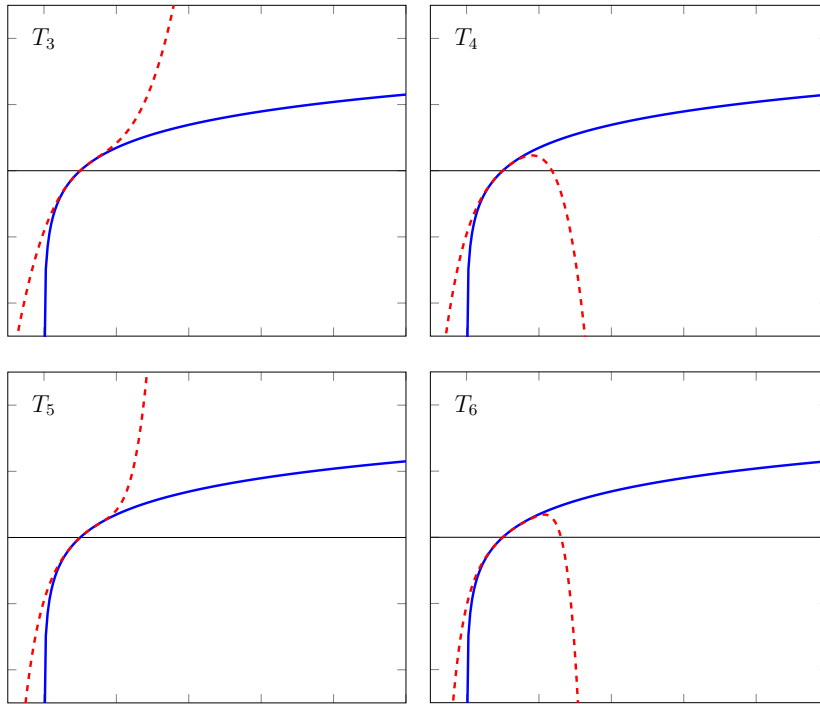
To get you started (you will not be given this on the quiz), here is the degree 2 Taylor polynomial:

$$T_2(x) = 0 + (x - 1) + \frac{-1}{2}(x - 1)^2.$$

And, for fun, here are graphs of various Taylor polynomials for  $\ln(x)$  at  $a = 1$ , graphed along with  $\ln(x)$ :







**Also for fun:** Check out this website to have fun with Taylor polynomials for different functions:

<https://www.geogebra.org/m/s9SkCsvC>.