Lecture 35

Exponential growth and decay

We are now familiar with derivatives and integrals.

The rest of the class will be about applications of these ideas—either to the sciences, or to other mathematically interesting problems.

It is often the case that the amount of something not only changes, but changes *proportionally* to how much something there *is*.

35.1 Example: Viruses

For example, suppose that $P(t)$ is a function telling you how many people are infected with a virus at time t. (Let's say t is in hours, so $P(t)$ is how many people are infected at time t hours. For concreteness, we can let $t = 0$ correspond to 12:01 AM of January 1, 2020.)

How would you expect $P(t)$ to be *changing*, especially if the virus is brand new?

A naive model might assume that every infected person, on average, infects three new people per hour. (This would be an *incredibly* contagious virus.)

$P(t)$ (The number of people infected)	$\frac{dP}{dt}$ (The rate at which new people are being infected)
1	3 per hour
2	6 per hour
1000	3000 per hour

Note that the numbers in the righthand column are proportional to the numbers in the lefthand column. In fact, if we assume that every infected person is infecting 3 new people per hour, we always have

$$\frac{dP}{dt} = 3P(t).$$

Can you find a function $P(t)$ such that this equation (relating the derivative of P to P itself) holds?

Note that if we didn't assume that each infected person can cause 3 new infections per hour, but say, 0.1 new infections per hour, then we would find that

$$\frac{dP}{dt} = 0.1P(t).$$

More generally, if k is how many new people one infected person can infect per hour, we have

$$\frac{dP}{dt} = kP(t). \tag{35.1.1}$$

The bigger the k, the more infections the virus.

Remark 35.1.1. There is something that becomes unrealistic about the assumption that *every* infected person can infect 3 new people per hour. For example, there are only so many people in the world! And if all the infected people are clustered together, they don't have much of a chance to spread the virus to new people either—on the other hand, the larger the number of infected people, the more likely that clusters will form, slowing new infections. So *eventually* there's no way that a virus can sustain this growth of 3 new infections per infected person. But, *initially*, it's not a bad model.

Remark 35.1.2. The above model doesn't have to be about viruses; it can be about *any* population of living things that can grow without restrictions. For example, if you assume that, on average, each human creates about 1 new human every 40 years (so about 0.025 new humans per year) the human population $P(t)$ at time t (where t is in years) would be be a function such that

$$\frac{dP}{dt} = 0.025P(t).$$

35.2 Example: Growth of money

Most savings accounts will give you interest. For example, if you have an annual interest of 0.2 percent on your savings account[1], and if you have B dollars over the

[1] For savings accounts, interest can typically range from somewhere between 0.01 and 2 percent. Big banks tend to have worse rates, while credit unions and online-only accounts tend to have the

course of a year in that account, your bank will then give you (for free!) 0.2 percent of B dollars. That is, after a year, the amount of money in your account will be

$$B + (0.002)B.$$

This isn't very much money. Even if you had 10,000 dollars in your account (which is a lot!) interest gained would only be 20 dollars over the course of the year.[2]

If you don't deposit any new money into your savings account, how is your account balance changing? Very roughly speaking, it's changing at about $(0.002)B$ per year, where B is the amount of money that's actually in there over a year. Thus,

$$\text{annual rate of change of } B(t) = 0.002B(t).$$

A year is a long time. To be able to do calculus, we want "instantaneous" rates of change, so we may want to consider a model that tells us how much our bank balance is changing per day, or per hour, or per minute (or better: per microsecond). And, in fact, it is quite convenient to model savings accounts as though they *continuously* give you interest, so that

$$\text{instantaneous rate of change of } B(t) = kB(t)$$

for some "instantaneous" interest rate k. (It turns out that if your annual interest rate is R, then $k = \ln(1 + R)$. So for example, our example had $R = 0.002$, so $k = \ln(1.002)$. This is about 0.0019, so only a little less than R itself.) Writing the righthand side in more familiar notation, we have that

$$\frac{dB}{dt} = kB(t). \qquad (35.2.1)$$

35.3 Example: Radioactive decay

Radioactive material loses mass—this is because radioactive material turn into radiation.

Now, how much radioactive material is lost/radiated? Well, if a 5 gram sample is losing mass at about 1 gram per year, then a 10 gram sample would lose mass at 2 grams per year. In this example,

$$\text{Rate of change of mass} = 0.2\text{mass}.$$

best rates.

[2]By the way, money devalues as a result of inflation, and the interest of most savings accounts does not keep up with inflation; meaning that while you might feel like you're making money, your money is actually losing value by just sitting and accruing interest in most savings accounts.

Or, if $M(t)$ is a function for how many grams of (for example) Uranium is in a sample at time t, we have that

$$\frac{dM}{dt} = kM(t) \tag{35.3.1}$$

for some constant t. Note that because mass is *decreasing*, k is now a negative constant (unlike the previous examples).

35.4 The differential equation

All three examples are important. The virus example is obviously salient these days, but more generally, it is important to be able to model population growth for various species (and certainly for humans). Wouldn't you like to be able to predict the population of the world, or of the country, ten or fifteen years out from now?

The money example is important because you might want to plan for retirement with some idea of how much your money will have grown by then.

The radioactive decay example is also important—this is how we use carbon dating to determine the age of organic samples, for example. (How do you think we know how old certain samples are?)

The beauty is that they are all modeled by the same kind of equation:

$$\frac{df}{dt} = kf(t) \tag{35.4.1}$$

where f is some function we're interested in.

Exercise 35.4.1. Can you think of a function $f(t)$ that satisfies Equation 35.4.1?

35.5 The solution

You can take the following theorem for granted:

Theorem 35.5.1. If f is a function whose derivative satisfies Equation (35.4.1), then f must be a function of the form

$$f(t) = Ae^{kt}$$

where $A = f(0)$. (Note that k is the same constant as in (35.4.1).)

The theorem is great because it immediately gives mathematical models for all the three problems we were interested in.

35.6 Modeling virus spread (in early stages)

Let's say you know that there are 5 people infected with a disease on day 0, but you are not sure about the infection rate of the disease. We can see how quickly this virus grows by using different values of k. (Roughly speaking, if you think that each infected person produces R new infections every day, it turns out you could model k by $\ln(1+R)$.)

Table 35.1 is a table of the values

$$f(t) = 5e^{kt}$$

for various values of k.

Very, very roughly, this models the growth of a disease wherein an infected person infects about k humans per day. This heuristic becomes a little inaccurate for larger values of k (like 0.7), but that's okay.

As you can see, changing k makes a huge difference in how many people are infected by day 7, by day 14, by day 21. I hope it's sobering to think that, in a span of 14 days (e.g., this year's Spring Break) changing the value of k can result in either 20 people being infected, or 5,000 people. If k is big enough, the infection numbers become (unrealistically) gargantuan after 31 days.

I also want to emphasize that this k is not some number inherent to the virus; by changing our behaviors, we can change (on average) how many new infections one infected person can cause. (For example, if you are an infected person, you could choose to stay at home, or you could choose to go to a packed dance party every night. If every infected person made the right choice, k would be small. If everybody made the wrong choices, k would be large.)

Finally, this also illustrates why delayed public policies—even if delayed by just one week—can have disastrous consequences. In only the first seven days in the table, good behavior might lead to only 10 total infections, while bad behavior might lead to 166, or 671 total infections. To put things in perspective: The United States has known about this virus since January. States like Alabama (whose infection rates per capita rival those of Washington State and New York) did not implement a stay-at-home order until April.

Remark 35.6.1 (This model is only good for the early stages of an outbreak)**.** The table also shows some shortcomings of this model. For instance, when $k = 0.7$, the model predicts nearly 7 billion infections by day 30. The current world population is about 7.8 billion. Doesn't that seem crazy?

Day	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7** (k values)
0	5	5	5	5	5	5	5	5
1	5	6	6	7	7	8	9	10
2	5	6	7	9	11	14	17	20
3	5	7	9	12	17	22	30	41
4	5	7	11	17	25	37	55	82
5	5	8	14	22	37	61	100	166
6	5	9	17	30	55	100	183	333
7	5	10	20	41	82	166	333	671
8	5	11	25	55	123	273	608	1,352
9	5	12	30	74	183	450	1,107	2,723
10	5	14	37	100	273	742	2,017	5,483
11	5	15	45	136	407	1,223	3,675	11,042
12	5	17	55	183	608	2,017	6,697	22,235
13	5	18	67	247	906	3,326	12,203	44,776
14	5	20	82	333	1,352	5,483	22,235	90,169
15	5	22	100	450	2,017	9,040	40,515	181,578
16	5	25	123	608	3,009	14,905	73,824	365,652
17	5	27	150	820	4,489	24,574	134,516	736,333
18	5	30	183	1,107	6,697	40,515	245,104	1,482,793
19	5	33	224	1,494	9,991	66,799	446,609	2,985,978
20	5	37	273	2,017	14,905	110,132	813,774	6,013,021
21	5	41	333	2,723	22,235	181,578	1,482,793	12,108,738
22	5	45	407	3,675	33,171	299,371	2,701,825	24,384,004
23	5	50	497	4,961	49,486	493,579	4,923,046	49,103,355
24	5	55	608	6,697	73,824	813,774	8,970,374	98,882,013
25	5	61	742	9,040	110,132	1,341,686	16,345,087	199,123,922
26	5	67	906	12,203	164,298	2,212,067	29,782,690	400,986,337
27	5	74	1,107	16,472	245,104	3,647,082	54,267,599	807,487,322
28	5	82	1,352	22,235	365,652	6,013,021	98,882,013	1,626,079,781
29	5	91	1,651	30,015	545,489	9,913,796	180,174,775	3,274,522,561
30	5	100	2,017	40,515	813,774	16,345,087	328,299,846	6,594,078,672
31	5	111	2,464	54,690	1,214,008	26,948,492	598,201,321	13,278,843,780

Table 35.1: A table of the values of $f(t) = 5e^{kt}$ for various k and various t (t is measured in days).

Indeed, no virus, and no population, can grow unfettered. A real model should account for certain constraints (like a virus can't infect more people than the number of people on earth). We'll see one way to do this next class.

Let me just say that this model we're using is useful for the *early* stages of a virus, when there is ample opportunity to spread and there is reason to expect that k does not experience huge variability.

35.7 Exponential growth

The function

$$e^t$$

and its relatives,

$$f(t) = Ae^{kt}$$

grow *very, very* fast (when $k > 0$). By grow, I mean that if t is large enough, then small changes in t can result in huge changes in the value $f(t)$.

Let's compare how Ae^{kt} grows to how, for example, $g(t) = 5t$ grows. No matter how big t is, if I change t by 0.1, $g(t)$ changes value by $5 \times 0.1 = 0.5$. However,

$$\begin{aligned}
f(t+0.1) - f(t) &= Ae^{k(t+0.1)} - Ae^{kt} && (35.7.1)\\
&= Ae^{kt+0.1k} - Ae^{kt} && (35.7.2)\\
&= Ae^{kt}e^{0.1k} - Ae^{kt} && (35.7.3)\\
&= Ae^{kt}(e^{0.1k} - 1) && (35.7.4)\\
&= f(t)(e^{0.1k} - 1). && (35.7.5)
\end{aligned}$$

Thus, if t is large enough, then the difference between $f(t+0.1)$ and $f(t)$ is large as well. You can certainly expect the difference to be larger than 0.5!

In fact, exponential functions like $f(t) = Ae^{kt}$ will eventually grow faster than any polynomial. For example,

$$\lim_{t\to\infty} \frac{Ae^{kt}}{x^{10}} = \infty$$

(and this limit is ∞ regardless of what polynomial we put in the denominator). That this limit is ∞ means tells us that the numerator function grows faster than the denominator function as t gets big.

In everyday language, "exponential" is used to me "really quick" or "really big." In math, to grow exponentially means to grow as fast as Ae^{kt} does (for some k).

More rigorously, we say that a function g grows exponentially if there is some A and k such that

$$\lim_{x\to\infty} \frac{Ae^{kt}}{g(t)}$$

is finite.

In the zoo of familiar functions, you should think of these Ae^{kt} as very, very quickly growing animals.

Remark 35.7.1. Note that if $k = \ln 2$, we have that

$$Ae^{kt} = Ae^{\ln 2t} = A(e^{\ln 2})^t = A2^t.$$

So in fact, any function of the form

$$Ab^t$$

(note that the variable t *is* the exponent, while A and b are just constants) is thus an "exponential" function.

35.8 Preparation for next time

I want you to upload solutions to the following problems by 10 PM on Monday, April 13.

35.8.1

Let $f(t) = Ae^{kt}$. (As before, A and k are just constant real numbers. We will assume both A and k are positive.)

(a) If t is in units of days, how many days does it take for $f(t)$ to *double* in value? Your answer should be in terms of k and some other functions and numbers you know.

(b) At what value of t do we have that $f(t) = 2f(3)$?

35.8.2

Let $M(t)$ be the mass of some radioactive material after t years of radioactivity. As we saw before, we know that

$$M(t) = Ae^{kt}$$

for some *negative* k, while $A = M(0)$ is the mass of the material at 0 years' time.

(a) In terms of k, how many years does it take for $M(t)$ to *halve* in value? (This is called the *half-life* of the radioactive material.)

(b) At what value of t will we have that $M(t) = \frac{1}{2}M(3)$?

35.8.3

At $t = 0$, Rafael invests 100 dollars into an index fund. The value of Rafael's investment is modeled by the function

$$f(t) = 100e^{kt}$$

where f is in dollars and t is in years.

(a) Assume Rafael is 30 years old at $t = 0$. If $k = \ln(1 + 0.05)$ (this models about 5 percent annual interest), how much money does Rafael have at the age of 65, when he retires? You can use a calculator to round to the nearest dollar.

(b) Rafael's friend, Susana, also invests 100 dollars into a fund with $k = \ln(1+0.05)$. But she does this at the age of 35. How much will her investment be worth when Susana is 65 years old?

(c) How much of a difference, in dollars, did waiting 5 years make?