Lecture 34

L'Hôpital's rule

Exercise 34.0.1. Remember that

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1.$$

For no good reason, let's take the derivative of the top and bottom functions, and then take the limit:

$$\lim_{x \to 0^+} \frac{(\sin(x))'}{(x)'}.$$

What answer do you get?

Exercise 34.0.2. Compute the limit

$$\lim_{x \to \infty} \frac{2x+3}{5x-7}.$$

Let's try taking the derivative of the top and bottom function first, and then take the limit. That is, compute

$$\lim_{x \to \infty} \frac{(2x+3)'}{(5x-7)'}.$$

How do your answers compare?

Exercise 34.0.3. Compute the limits

$$\lim_{x \to \infty} \frac{x^2}{1/x} \qquad \text{and} \qquad \lim_{x \to \infty} \frac{(x^2)'}{(1/x)'}.$$

How do your answers compare?

The first two exercises were promising, but the last one showed that this trick doesn't always work. Here is a theorem that you may use freely; we won't prove it in this class:

Theorem 34.0.4 (L'Hôpital's Rule)**.** Let f and g be functions. If

1. $\lim f(x) = \infty$ and $\lim g(x) = \infty$, or if
2. $\lim f(x) = 0$ and $\lim g(x) = 0$,

then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

provided the righthand side exists.

In words, L'Hôpital's Rule says that: If the limits of f and g are both some sort of infinity, or both zero, then the limit of the fraction may be computed by first taking the derivatives of f and g (so long as the limit of the derivatives exist).

Remark 34.0.5. Some textbooks use instead the condition $\lim |f(x)| = \infty$ (and likewise for g), which can seem a little confusing. It turns out this condition is identical to "$\lim f(x) = \infty$ or $\lim f(x) = -\infty$" and likewise for g. (In general, it is very rare for a function to satisfy $\lim |f(x)| = \infty$ without satisfying $\lim f(x) = \infty$ or $\lim f(x) = -\infty$. In fact, such a scenario is impossible if f is continuous and defined for all large values of x. And secretly, we are assuming that f is defined for all large values of x when we compute limits of f as x approaches infinity.)

In either case, these cases all follow from what we've stated above. For example, by the scaling law, $\lim f(x) = -\lim(-f(x))$, so we can always convert a limit equaling $-\infty$ to one equaling ∞.

Warning 34.0.6. The hypothesis of L'Hôpital's Rule is important! (The limits of the denominator and numerator must both agree.) You saw in Exercise 34.0.3 an example where the numerator and denominator had different limits; as a result, the limit of the fraction after taking the derviatives was *different* from the limit of the fraction.

It may also be that the limit of f/g exists, but the limit of f'/g' doesn't exist. Example: $f(x) = x + \cos x$ and $g(x) = x$ and the limit as $x \to \infty$.

Remark 34.0.7. The limits in the statement of L'Hôpital's Rule have no subscripts. This is because I am being lazy. To be explicit: If all the limits are taken at the same point, then the theorem holds.

For example, if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^+} g(x)$ both equal zero, you can apply L'Hôpital's Rule:

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

This works for one-sided limits from the left, and for limits at $\pm\infty$.

Remark 34.0.8. As you may have guessed, "L'Hôpital" is a French name. It is pronounced (roughly) "Lo-pee-tahl." You may not be used to the ô; that is, to the little "hat" on top of the *o*. This symbol is called a *circumflex,* and in French, it is often used when a word *used* to have an *s* right after the circumflex. So for example, in the past, the word "L'Hôpital" would have been spelled "L'Hospital." Yes, that's right; this person's name literally translates to "The Hospital."

Exercise 34.0.9. Evaluate the following limits. Some may involve L'Hôpital's rule; other may not. When you use L'Hôpital's rule, say why you know you can use it (based on the hypotheses of the theorem above).

(a) $\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)}$

(b) $\lim_{x\to\infty} \frac{x}{x^2-1}$

(c) $\lim_{x\to 1^+} \frac{x}{x^2-1}$

(d) $\lim_{x\to-\infty} \frac{1}{2x+3}$

(e) $\lim_{x\to 0^+} x\ln x$

(f) $\lim_{x\to 0^-} \frac{x}{\sin(x)}$

(g) $\lim_{x\to\infty} xe^x$

(h) $\lim_{x\to-\infty} xe^x$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

34.1 For next time

You should be able to compute all the limits above (and limits similar to them).

(Solutions are on next page if necessary.)

Solutions

(a) $\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)}$

Evaluating the limit in the numerator and denominator yields

$$\frac{\lim_{x\to(\pi/2)^+}(x-\pi/2)\sin(x)}{\lim_{x\to(\pi/2)^+}\cos(x)} = \frac{(\pi/2-\pi/2)\cdot 1}{0} \tag{34.1.1}$$

This is 0/0, so we can use L'Hôpital's rule.

$$\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)} = \lim_{x\to(\pi/2)^+} \frac{((x-\pi/2)\sin(x))'}{(\cos(x))'} \tag{34.1.2}$$

$$= \lim_{x\to(\pi/2)^+} \frac{\sin(x)+(x-\pi/2)\cos(x)}{-\sin(x)} \tag{34.1.3}$$

$$= \lim_{x\to(\pi/2)^+} \frac{1+0}{-1} \tag{34.1.4}$$

$$= -1. \tag{34.1.5}$$

(b) $\lim_{x\to\infty} \frac{x}{x^2-1}$

Evaluating limits in the numerator and denominator, we obtain ∞/∞, so we can use L'Hôpital's rule.

$$\lim_{x\to\infty} \frac{x}{x^2-1} = \lim_{x\to\infty} \frac{(x)'}{(x^2-1)'} \tag{34.1.6}$$

$$= \lim_{x\to\infty} \frac{1}{2x} \tag{34.1.7}$$

$$= 0. \tag{34.1.8}$$

You also could have solved the original limit without L'Hôpital's Rule: Just divide top and bottom by x.

(c) $\lim_{x\to 1^+} \frac{x}{x^2-1}$

We *cannot* use L'Hôpital's Rule here because, when evaluating the limits of the numerator and denominator, we arrive at 1/0. This is not 0/0 nor ∞/∞.

But we can still divide top and bottom by x. Then

$$\begin{aligned}
\lim_{x\to 1^+} \frac{x}{x^2-1} &= \lim_{x\to 1^+} \frac{x}{x^2-1} \cdot \frac{1/x}{1/x} && (34.1.9)\\
&= \lim_{x\to 1^+} \frac{1}{x - \frac{1}{x}} && (34.1.10)\\
&= \frac{\lim_{x\to 1^+} 1}{\lim_{x\to 1^+} x - \frac{1}{x}} && (34.1.11)\\
&= \frac{1}{\lim_{x\to 1^+} x - \frac{1}{x}}. && (34.1.12)
\end{aligned}$$

When $x > 1$, we know that $x - 1/x$ is positive. So the denominator approaches 0 from the right.

$$= \frac{1}{0^+} = \infty.$$

Here is another way you could have computed this limit. Note that $(x^2 - 1) = (x + 1)(x - 1)$, and we know that $(x - 1)$ is the factor that is causing the denominator to become 0 in the limit. So let's rewrite things in a way we can try to factor out an $(x - 1)$ from the *numerator*, too:

$$\begin{aligned}
\lim_{x\to 1^+} \frac{x}{x^2-1} &= \lim_{x\to 1^+} \frac{x}{(x-1)(x+1)} && (34.1.13)\\
&= \lim_{x\to 1^+} \frac{x-1+1}{(x-1)(x+1)} && (34.1.14)\\
&= \lim_{x\to 1^+} \frac{x-1}{(x-1)(x+1)} + \lim_{x\to 1^+} \frac{1}{(x-1)(x+1)} && (34.1.15)\\
&= \lim_{x\to 1^+} \frac{1}{x+1} + \lim_{x\to 1^+} \frac{1}{(x-1)(x+1)} && (34.1.16)\\
&= \lim_{x\to 1^+} \frac{1}{1} + \lim_{x\to 1^+} \frac{1}{x^2-1} && (34.1.17)\\
&= 1 + \lim_{x\to 1^+} \frac{1}{x^2-1}. && (34.1.18)
\end{aligned}$$

Now note that $x^2 - 1$ approaches 0 from the right when $x \to 1^+$, because if $x > 1$, then $x^2 > 1$. So this limit becomes

$$= 1 + \frac{1}{0^+} = 1 + \infty = \infty$$

just as before.

(d) $\lim_{x\to-\infty}\frac{1}{2x+3}$ We don't need L'Hôpital's Rule: We see this limit is $1/-\infty = 0$. Note that we couldn't have used L'Hôpital's Rule anyway.

(e) $\lim_{x\to 0^+} x\ln x$

Let's rewrite this limit:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \tag{34.1.19}$$

$$= -\lim_{x\to 0^+}\frac{-\ln x}{1/x}. \tag{34.1.20}$$

You don't really need to insert the minus sign, but I did so to see that $\lim_{x\to 0^+} -\ln x = \infty$ and $\lim_{x\to 0^+} = \infty$; this shows we can apply L'Hôpital's Rule. Applying said rule, we find

$$-\lim_{x\to 0^+}\frac{-\ln x}{1/x} = -\lim_{x\to 0^+}\frac{(-\ln x)'}{(1/x)'} \tag{34.1.21}$$

$$= -\lim_{x\to 0^+}\frac{-1/x}{-1/x^2} \tag{34.1.22}$$

$$= -\lim_{x\to 0^+}\frac{1/x}{1/x^2} \tag{34.1.23}$$

$$= -\lim_{x\to 0^+}\frac{1/x}{1/x^2}\cdot\frac{x^2}{x^2} \tag{34.1.24}$$

$$= -\lim_{x\to 0^+}\frac{x}{1} \tag{34.1.25}$$

$$= -\frac{0}{1} \tag{34.1.26}$$

$$= 0. \tag{34.1.27}$$

(f) $\lim_{x\to 0^-}\frac{x}{\sin(x)}$

We already know that $\lim_{x\to 0}\frac{\sin x}{x} = 1$, so we can use that to say

$$\lim_{x\to 0^-}\frac{x}{\sin(x)} = \lim_{x\to 0^-}\frac{1}{\frac{x}{\sin(x)}} \tag{34.1.28}$$

$$= \frac{1}{\lim_{x\to 0^-}\frac{x}{\sin(x)}} \tag{34.1.29}$$

$$= \frac{1}{1} \tag{34.1.30}$$

$$= 1. \tag{34.1.31}$$

You could also use L'Hôpital's Rule to arrive at

$$\lim_{x\to 0^+} \frac{1}{\cos(x)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$$

(g) $\lim_{x\to\infty} xe^x$ You don't need L'Hôpital's Rule here; we plainly see that the limit is given by $\infty \cdot \infty = \infty$.

(h) $\lim_{x\to -\infty} xe^x$ This gets trickier, because we find (taking naive limits)

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} x \lim_{x\to-\infty} e^x = (-\infty)\cdot(0)$$

which is undefined. So let's rewrite this limit as a fraction:

$$\lim_{x\to-\infty} xe^x = = \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{34.1.32}$$

$$= \frac{\lim_{x\to-\infty} x}{\lim_{x\to-\infty} e^x} \tag{34.1.33}$$

$$= -\frac{\lim_{x\to-\infty} -x}{\lim_{x\to-\infty} e^x} \tag{34.1.34}$$

$$= -\frac{\infty}{\infty} \tag{34.1.35}$$

which means we can use L'Hôpital's Rule on this limit. We find:

$$\lim_{x\to-\infty} xe^x = = \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{34.1.36}$$

$$= \frac{\lim_{x\to-\infty}(x)'}{\lim_{x\to-\infty}(e^x)'} \tag{34.1.37}$$

$$= \frac{\lim_{x\to-\infty} 1}{\lim_{x\to-\infty} e^x} \tag{34.1.38}$$

$$= \frac{1}{\lim_{x\to-\infty} e^x} \tag{34.1.39}$$

$$= \frac{1}{0^+} \tag{34.1.40}$$

$$= \infty. \tag{34.1.41}$$

$$\tag{34.1.42}$$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

Evaluating the numerator and denominator limits, we obtain ∞/∞, so we can use L'Hôpital's Rule. Then we end up with

$$\lim_{x\to\infty} \frac{\ln 5 5^x}{2x}.$$

Taking limits of top and bottom, again we find ∞/∞. So we can use L'Hôpital's Rule *again*. Then we find

$$\lim_{x\to\infty} \frac{(\ln 5)^2 5^x}{2}.$$

The limit of this expression is clearly ∞.

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

This problem is the same work as above, but you use L'Hôpital's Rule three times.