Lecture 33

Curve-sketching

Without using a graphing calculator, let's visualize the function

$$f(x) = \frac{x^2 + 1}{x^2 - 2}$$

using tools of calculus!

33.1 Asymptotes

First, let's find the vertical and horizontal asymptotes. Remember, you do this by computing

1. The limits at $\pm\infty$ (to find horizontal asymptotes), and
2. The limits where f looks undefined (there are vertical asymptotes if this limit is $\pm\infty$).

(1) You can compute that the horizontal asymptote is 1, and that f approaches 1 near both ∞ and $-\infty$. Here's the computation for the limit at $-\infty$; I'll leave the

other limit to you!

$$\lim_{x\to-\infty}\frac{x^2+1}{x^2-2} = \lim_{x\to-\infty}\frac{x^2+1}{x^2-2} \tag{33.1.1}$$
$$= \lim_{x\to-\infty}\frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}} \tag{33.1.2}$$
$$= \frac{\lim_{x\to-\infty}1+\frac{1}{x^2}}{\lim_{x\to-\infty}1-\frac{2}{x^2}} \tag{33.1.3}$$
$$= \frac{1+0}{1-0} \tag{33.1.4}$$
$$= 1. \tag{33.1.5}$$

(You *do* need to compute both limits, because there may be horizontal asymptotes with different heights.)

(2) f potentially has asymptotes where the denominator is zero—that is, when $x = \pm\sqrt{2}$. There are *four* one-sided limits to compute here. I will compute one for you, and tell you the answer for the rest:

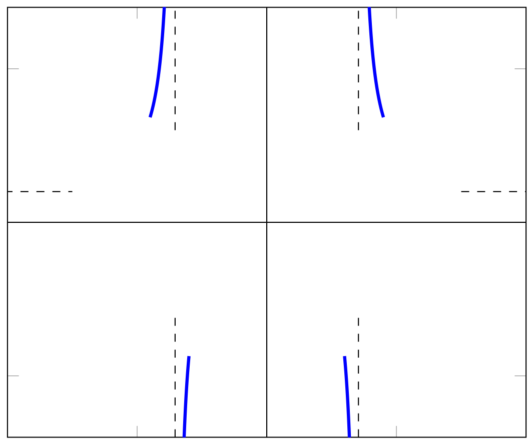
$$\lim_{x\to-\sqrt{2}^-}\frac{x^2+1}{x^2-2} = \lim_{x\to-\sqrt{2}^-}\frac{x^2+1}{x^2-2} \tag{33.1.6}$$
$$= \frac{\lim_{x\to-\sqrt{2}^-}x^2+1}{\lim_{x\to-\sqrt{2}^-}x^2-2} \tag{33.1.7}$$
$$= \frac{2+1}{0^+} \tag{33.1.8}$$
$$= \frac{3}{0^+} \tag{33.1.9}$$
$$= \infty. \tag{33.1.10}$$

The thing that requires most explanation is probably how we got to line (**??**). As x approaches $-\sqrt{2}$ from the left, x^2 approaches 2 *from the right*; that is, x^2 is shrinking toward 2. Thus, x^2-2 is approaching 0 from the right. This is why the denominator becomes 0^+.

The three other one-sided limits can be computed to be

$$\lim_{x\to-\sqrt{2}^+}\frac{x^2+1}{x^2-2} = -\infty, \qquad \lim_{x\to\sqrt{2}^-}\frac{x^2+1}{x^2-2} = -\infty, \qquad \lim_{x\to\sqrt{2}^+}\frac{x^2+1}{x^2-2} = \infty.$$

Based purely on these computations, we can begin to visualize the graph of $f(x)$:



Tbe dashed lines show where the asymptotes are; the thick blue lines are the beginnings of the graph. Note that I haven't yet drawn the parts of the graph near horizontal $\pm\infty$; this is because while I know the asymptotes, I don't know how f approaches the asymptotes (for example, f could oscillate close to the asymptote, or approach from above, or from below).

33.2 Concavity

Now I'd recommend computing the concavity of the function to get a good feel for shape.

Remember, concavity is dictated by whether the second derivative is positive or negative. So let's compute the second derivative.

First, we compute f':

$$\left(\frac{x^2+1}{x^2-2}\right)' = \frac{(x^2+1)'(x^2-2)-(x^2-2)'(x^2+1)}{(x^2-2)^2} \tag{33.2.1}$$
$$= \frac{(2x)(x^2-2)-(2x)(x^2+1)}{(x^2-2)^2} \tag{33.2.2}$$
$$= \frac{(2x)(x^2-2-(x^2+1))}{(x^2-2)^2} \tag{33.2.3}$$
$$= \frac{(2x)(-3)}{(x^2-2)^2} \tag{33.2.4}$$
$$= \frac{-6x}{(x^2-2)^2} \tag{33.2.5}$$

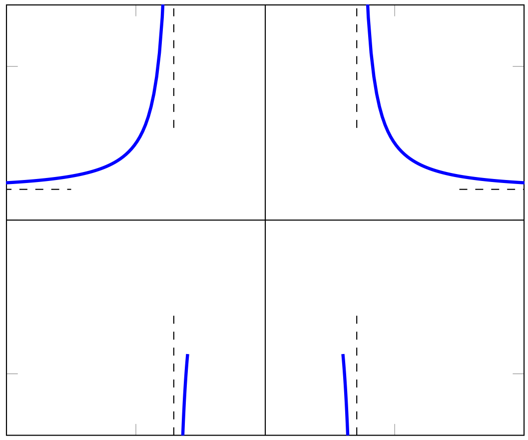
The second derivative is computed as follows:

$$\left(\frac{x^2+1}{x^2-2}\right)'' = \left(\frac{-6x}{(x^2-2)^2}\right)' \tag{33.2.6}$$
$$= \frac{(-6x)'(x^2-2)^2-(-6x)((x^2-2)^2)'}{(x^2-2)^4} \tag{33.2.7}$$
$$= \frac{(-6)(x^2-2)^2-(-6x)(x^2-2)(2x)}{(x^2-2)^4} \tag{33.2.8}$$
$$= \frac{(-6)(x^2-2)[(x^2-2)-(x)(2x)]}{(x^2-2)^4} \tag{33.2.9}$$
$$= \frac{(-6)(x^2-2)(-x^2-2)}{(x^2-2)^4} \tag{33.2.10}$$
$$= \frac{6(x^2-2)(x^2+2)}{(x^2-2)^4} \tag{33.2.11}$$

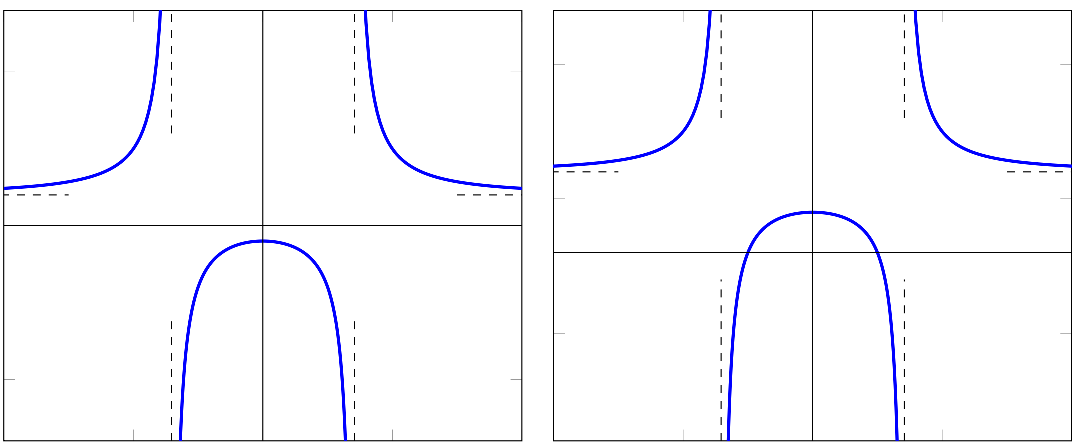
When is this fraction positive, and when is it negative?

The denominator, $(x^2-2)^4$, is always positive, so we can focus on the numerator, $6(x^2-2)(x^2+2)$.

(i) We see that x^2+2 is always positive, while x^2-2 is positive whenever $x^2>2$. That is, whenever $|x|>\sqrt{2}$. At this point, we know that the graph of the function must be concave up when $|x|>\sqrt{2}$. So we can begin to draw this portion of the graph:

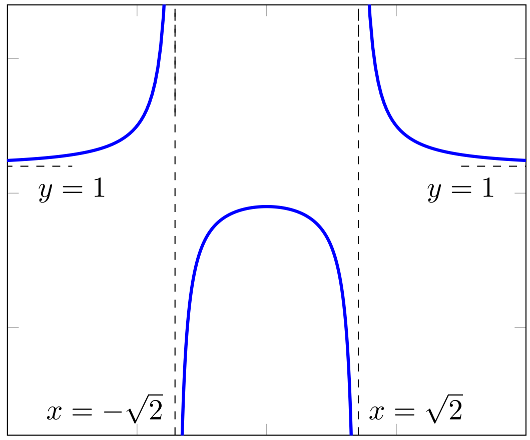


(ii) $6(x^2 - 2)(x^2 + 2)$ is negative precisely when $|x| < \sqrt{2}$, by the same reasoning as before; so we know that the function will look concave down in this region. So we can conclude the graph probably looks like one of the following:



At this point, there is still ambiguity in what the graph actually looks like. Where is the local maximum in the middle? At what x-coordinate? And at what y-coordinate?

But, depending on the kind of information you're looking for, you might be satisfied with a vague sketch as follows:



33.3 If you want more

If you want more information, or are asked for more information, you can make a more accurate sketch by finding out things such as:

- Identifying critical points.
- Finding the y- and x-intercepts.
- Labeling inflection points (in our case, we had none).

33.4 Summary and motivation

Let me emphasize one thing: **Your computations by hand are often more reliable than what graphing calculators will show you.** Being able to identify the critical points, the asymptotes, et cetera, can even tell you what frame you should use to look at a graph (e.g., what x values and what y values should your window hold?).

Here is a **summary of curve-sketching**: Identify the asymptotes, identify the concavity of the important regions, and then collect more information if you need (critical points, intercepts, et cetera).

Example 33.4.1. You are told that a function f has the following properties:

(a) $\lim_{x\to-\infty} f(x) = 3$.

(b) $\lim_{x\to\infty} f(x) = 2$.

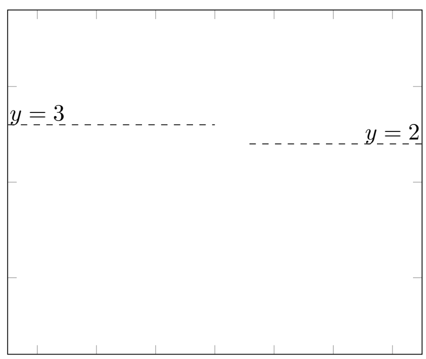
(c) f is continuous and defined everywhere.

(d) $f''(x)$ is positive when x is between -1 and 5

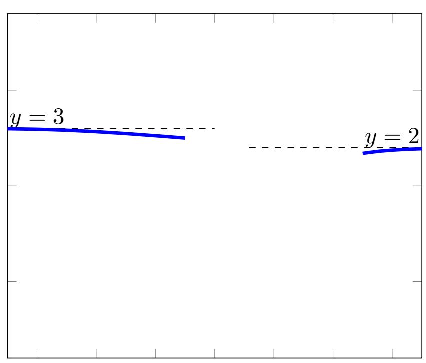
(e) $f''(x)$ is negative when $x < -1$ and when $x > 5$.

Sketch the graph.

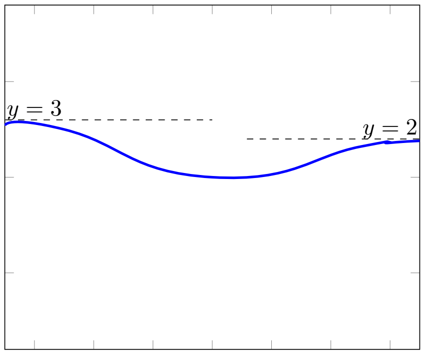
Solution: Based on (a) and (b), we can first draw the horizontal asymptotes, though we don't know how f approaches these asymptotes yet.



We know there are no vertical asymptotes by (c). By (e), we know that the function looks concave down outside of the interval $[-1, 5]$, so we can begin to draw as follows:



By (d), the rest of the function is concave up. So we sketch a "bowl up" shape:



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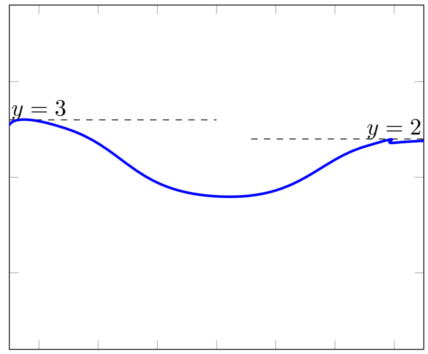


(With the information given, it is impossible to draw the graph of f with complete accuracy, but you see that you get a "feel" for what it looks like!)

33.5 For next time

For next time, I expect you to able to sketch the following graphs

(a) $f(x) = \frac{1}{x^2-2}$ (explaining *why* the sketch looks the way it does)

(b) $f(x) = \frac{1}{e^x-3}$ (explaining *why* the sketch looks the way it does)

(c) A continuous function f satisfying the following properties:

  (a) $\lim_{x\to\infty} f(x) = 5$
  (b) $\lim_{x\to-\infty} f(x) = -5$
  (c) $\lim_{x\to 2^+} f(x) = \infty$
  (d) $\lim_{x\to 2^-} f(x) = \infty$
  (e) $f''(x) < 0$ when x is less than -10,
  (f) $f''(x) > 0$ when x is between -10 and 2,
  (g) $f''(x) > 0$ when x is larger than 2.

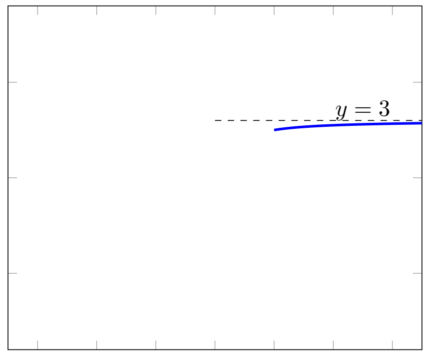
33.6 Appendix: Concavity near ∞

As you get to sketching many curves, it's natural to ask the following question: If I know

- $\lim_{x\to\infty} f(x) = 3$ and
- $f''(x) < 0$,

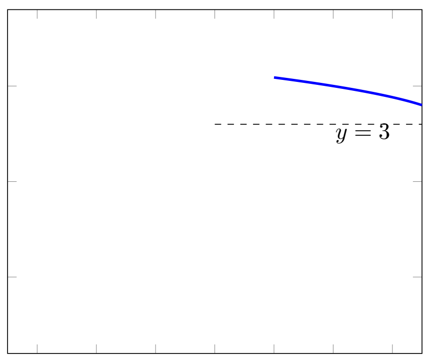
why do I know that f has to look like

DRAWING I:



near ∞? For example, why couldn't f look like the following?

DRAWING II:



In this appendix, I claim: **Drawing II can never happen** while f satisfies the two properties above. That is, if f is concave down and has a horizontal asymptote, f must approach that asymptote from *underneath* the asymptote, not from above.

Here is a great place for *proof*. Let's try to put into mathematical language what picture you're drawing in Drawing II: You seem to be drawing a function with

1. $f'' < 0$ for x larger than (for example) 5, and
2. $f' < 0$ for x larger than 5.

I want to emphasize that the role of 5 could be swapped with any number; so let's just call that number a from now on.

I claim the following:

Proposition 33.6.1. Suppose that f is a function such that, for some real number a, we have

1. $f''(x) < 0$ for all $x > a$, and
2. $f'(x) < 0$ for all $x > a$.

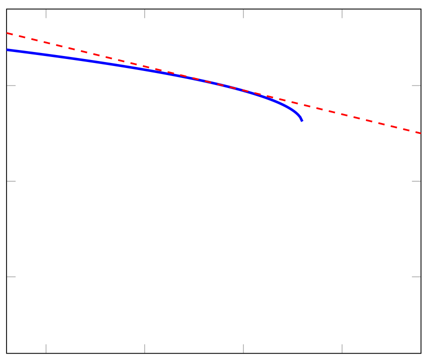
Then $\lim_{x\to\infty} f(x) = -\infty$. In particular, f does not have a horizontal asymptote as $x \to \infty$.

Proof. Let b be any number bigger than a, and let's let $M = f'(b)$. (That is, M is the slope of the tangent line to f at b.)

Then we can define another function called

$$g(x) = M(x - b) + f(b).$$

The graph of g is a line—a line with slope M, and which passes through the point $(b, f(b))$.



Above, the graph of g has been represented as a dashed red line, and the graph of f as a solid blue curve. Note g has negative slope because of hypothesis 2. of the Proposition.

Now, because $f''(x) < 0$, we know that $f'(x)$ will be less than $f'(b)$ for all $x > b$. This is a consequence of the mean value theorem! It's because you can think of $h = f'(x)$ as a function; and when $h'(x) < 0$ for all $x > a$ (because of Hypothesis 1), we know that h is decreasing in value for all $x > a$, so $h(x)$ will be less than $h(b)$ whenever $x > b$.

Now, because $f'(x) < f'(b) = g'(x)$ for all $x > b$, we see that $f(x) < g(x)$ for all $x > b$. (This is an application of the mean value theorem again.)

So if $f(x) < g(x)$ for all $x > b$, we conclude that

$$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x).$$

But $g(x) = M \cdot (x - b) + f(b)$. So we see that

$$\lim_{x\to\infty} g(x) = M \cdot \left(\lim_{x\to\infty} (x-b)\right) + f(b) \tag{33.6.1}$$
$$= M \cdot (\infty - b) + f(b) \tag{33.6.2}$$
$$= M \cdot (\infty) + f(b) \tag{33.6.3}$$
$$= -\infty + f(b) \tag{33.6.4}$$
$$= -\infty. \tag{33.6.5}$$

(Note that $M \cdot \infty = -\infty$ because $M < 0$.)

Putting everything together, we see

$$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x) = -\infty$$

so $\lim_{x\to\infty} f(x) = -\infty$. □