Lecture 32

Limits at ∞

32.1 Examples

Exercise 32.1.1. For each of the following functions, determine whether $f(x)$ "approaches" a particular value as x becomes larger and larger. Drawing a rough sketch of the graph may be helpful.

(i) $f(x) = \frac{1}{x}$.

(ii) $f(x) = 2 + \frac{1}{x}$.

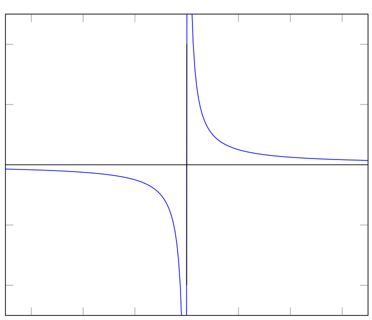
(iii) $f(x) = x$.

(iv) $f(x) = \sin x$.

(v) $f(x) = \frac{\sin x}{x}$.

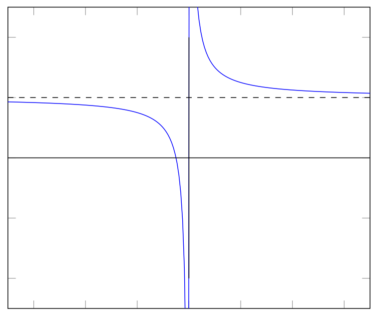
(vi) $f(x) = x \sin x$.

Here are the solutions.

(i) $f(x) = \frac{1}{x}$. 

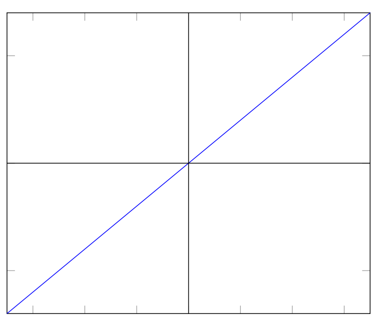
We see in this example that as x becomes bigger and bigger, $1/x$ becomes smaller and smaller; in fact, we can make $1/x$ as close to 0 as we like, so long as x is large enough. We write

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

(ii) $f(x) = 2 + \frac{1}{x}$. 

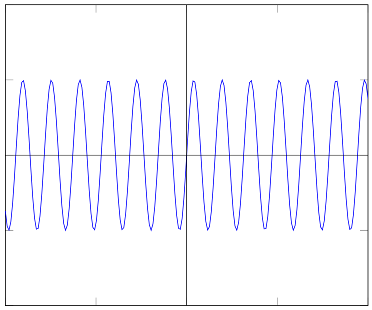
We see in this example that as x becomes bigger and bigger, $2 + 1/x$ becomes closer and closer to 2 (the height 2 is drawn in dashes above). In fact, we can make $2 + 1/x$ as close to 2 as we like, so long as x is large enough. We write

$$\lim_{x\to\infty} 2 + \frac{1}{x} = 0.$$

(iii) $f(x) = x$. 

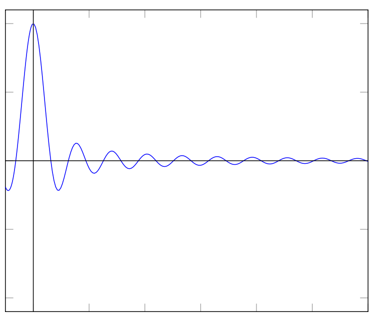
In this example, we see that $f(x) = x$ becomes bigger and bigger as x does. In fact, we can say the following: If we want f to be larger than some number T, we just need to ensure that x is larger than T. We say

$$\lim_{x\to\infty} x = \infty.$$

(iv) $f(x) = \sin x$. 

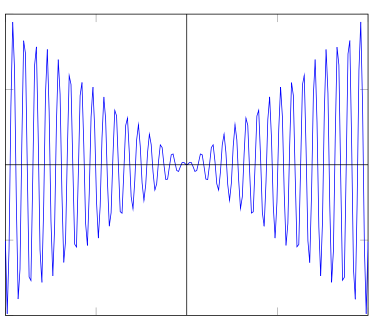
This is a tricky example, but we see that no matter how large x is, $f(x)$ could be any number between -1 and 1. And there is *no* big number that guarantees that "so long as x is bigger than this big number, $f(x)$ will be close to some limiting value." Thus, we say

$$\lim_{x\to\infty} \sin(x) \text{ does not exist }.$$

(v) $f(x) = \frac{\sin x}{x}$. 

This is different. f still seems to oscillate, but the f is approaching values closer and closer to 0 as x grows. Indeed, we can guarantee f to be ϵ-close to 0 so long as x is large enough. We say

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(vi) $f(x) = x \sin x$. 

In this example, $f(x)$ displays interesting behavior as x grows larger and larger. f oscillates, and more wildly. Importantly, f does *not* approach infinity as x grows. Here is why: To approach infinity, we must guarantee that for any T, f is larger than T so long as x is large enough. But regardless of how big we require x to be, there is a possibility that $f(x)$ is less than T—in fact, $f(x)$ could even be *negative*!

So we say

$$\lim_{x\to\infty} x\sin(x) \text{ does not exist.}$$

32.2 Definition of limits at infinity

We've seen some examples of limits at infinity. Here is a definition:

Definition 32.2.1. We say that f *has a limit at* ∞ if there exists a number L such that for every real number ϵ, we can guarantee that "if x is big enough, $f(x)$ is within ϵ of L."

More precisely, we say that f has a limit at ∞ if there exists a number L such that for every real number ϵ, we can find a number F so that[1]

$$x > F \implies |f(x) - L| < \epsilon. \tag{32.2.1}$$

(Remember that " $\implies$ " means "implies.")

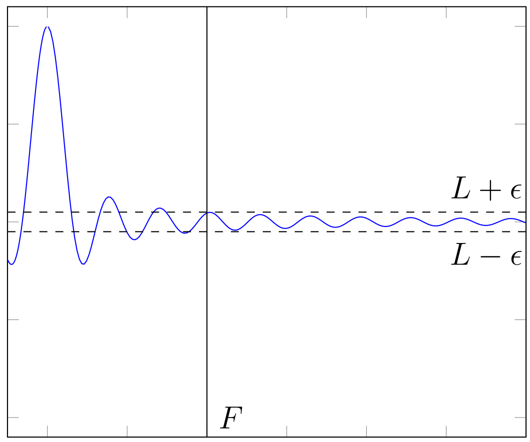
We call L the *limit of f as x approaches* ∞, and we write

$$\lim_{x\to\infty} f(x) = L.$$

Graphically, (32.2.1) means that so long as our x coordinate is larger than F, our

[1]Here, $x > F$ is the mathematical translation of "x is big enough."

graph of f is within a strip of height 2ϵ centered at L:



You will rarely have to use this definition, but you should know that the definition above provides the mathematical precision necessary to prove things like limit laws for infinity (see next section).

We can also talk about **limits as x approaches** $-\infty$—to find such limits is to ask whether f approaches a particular number as x becomes more and more negative. We write such a limit as

$$\lim_{x\to-\infty} f(x).$$

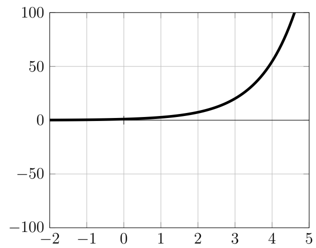
Remark 32.2.2 (Are there no "one-sided" limits at infinity?)**.** You may have noticed we have not dicussed one-sided limits when we approach ∞ or $-\infty$. A better way to think about this is that all limits at ∞ are in some sense one-sided, in that

$$\lim_{x\to\infty} f = \lim_{x\to\infty^-} f.$$

Indeed, there is no sense in which x can approach ∞ "from the right." Likewise, you should think of a limit at $-\infty$ as one-sided, too:

$$\lim_{x\to\infty} f = \lim_{x\to\infty^+}.$$

Example 32.2.3. This is an **important example you need to know**. Look at the graph of e^x:



As x approaches $-\infty$ (i.e., as x goes to the left), the graph approaches the x-axis. So

$$\lim_{x\to-\infty} e^x = 0.$$

As x approaches ∞ (i.e., as x goes to the right), the function grows larger and larger, without bound. So

$$\lim_{x\to\infty} e^x = \infty.$$

32.3 Practice with limit laws for infinities

We will be vague about this, but here it is:

Limit laws work for limits involving ∞^*

with an asterisk: $*$ **so long as all terms are defined**.

Example 32.3.1. Compute

$$\lim_{x\to\infty}(x - x^2).$$

We can try using the addition law. If we do, we find

$$\lim_{x\to\infty}(x - x^2) = = \lim_{x\to\infty} x - \lim_{x\to\infty} x^2 \tag{32.3.1}$$

$$= \infty - \infty \qquad \text{!!!!!!!} \tag{32.3.2}$$

The big exclamation marks are a warning: The expression "$\infty - \infty$" is **not defined**. This means that the limit law gives us no information (just like the quotient law is inapplicable when the denominator has limit 0). So we tried, and we failed. That's okay.

Let's try something else: The product law. The key observation is to see that

$$(x - x^2) = x(1 - x).$$

Then we have:

$$\begin{aligned}
\lim_{x\to\infty}(x-x^2) &= \lim_{x\to\infty} x(1-x) && (32.3.3)\\
&= \lim_{x\to\infty} x \cdot \lim_{x\to\infty}(1-x) && (32.3.4)\\
&= \infty \cdot \lim_{x\to\infty}(1-x) && (32.3.5)\\
&= \infty \cdot (1-\infty) && (32.3.6)\\
&= \infty \cdot (-\infty) && (32.3.7)\\
&= -\infty. && (32.3.8)
\end{aligned}$$

Example 32.3.2. Compute

$$\lim_{x\to\infty}(x-x^2+10).$$

We can try using the addition law. If we do, we find

$$\begin{aligned}
\lim_{x\to\infty}(x-x^2+10) = &= \lim_{x\to\infty}(x-x^2) + \lim_{x\to\infty} 10 && (32.3.9)\\
&= (\lim_{x\to\infty}(x-x^2)) + 10. && (32.3.10)
\end{aligned}$$

But we know this limit in the parentheses! We saw above that the limit was $-\infty$, so we obtain

$$\lim_{x\to\infty}(x-x^2+10) = -\infty + 10 = -\infty.$$

32.4 Limits at $\pm\infty$ for polynomials

In fact, repeating the factoring trick and the addition law, you can conclude the following: **You can compute limits at ∞ for polynomials by looking at the highest degree term**, and these limits will always be $\pm\infty$. For example,

$$\begin{aligned}
\lim_{x\to\infty} 3x^5+x^4-3x^2 &= \lim_{x\to\infty} 3x^5 && (32.4.1)\\
&= 3(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x)\cdot(\lim_{x\to\infty} x) && (32.4.2)\\
&= 3\infty\cdot\infty\cdot\infty\cdot\infty\cdot\infty && (32.4.3)\\
&= 3\infty && (32.4.4)\\
&= \infty. && (32.4.5)
\end{aligned}$$

The first equality is using the bolded principle above (you need only look at the highest degree term of the polynomial when computing limits at $\pm\infty$). The next line is using the product rule a lot.

As another example,

$$\lim_{x\to-\infty} 2x^6 + x^5 - 3x = \lim_{x\to-\infty} 2x^6 \tag{32.4.6}$$
$$= 2(\lim_{x\to-\infty} x)^6 \tag{32.4.7}$$
$$= 2(-\infty)^6 \tag{32.4.8}$$
$$= 2\infty \tag{32.4.9}$$
$$= \infty. \tag{32.4.10}$$

Again, in the first line, I am using that the limit at $\pm\infty$ of a polynomial is equal to the limit of the highest degree term. Note that I got lazy and wrote $(-\infty)^6$ rather than $(-\infty)\cdot(-\infty)\cdot(-\infty)\cdot(-\infty)\cdot(-\infty)$.

Here is one more examples for your edification:

$$\lim_{x\to-\infty} -4x^3 + x^2 - x = \lim_{x\to-\infty} -4x^3 \tag{32.4.11}$$
$$= -4\cdot\lim_{x\to-\infty} x^3 \tag{32.4.12}$$
$$= -4\cdot(-\infty)^3 \tag{32.4.13}$$
$$= -4\cdot(-\infty) \tag{32.4.14}$$
$$= \infty. \tag{32.4.15}$$

32.5 Limits at $\pm\infty$ for rational functions

Example 32.5.1. Let's compute

$$\lim_{x\to\infty} \frac{x^3+x+1}{3x^3-3x^2}.$$

Let's try using the quotient law. We get

$$\lim_{x\to\infty} \frac{x^3+x+1}{3x^3-3x^2}. = \frac{\lim_{x\to\infty} x^3+x+1}{\lim_{x\to\infty} 3x^3-3x^2} \tag{32.5.1}$$
$$= \frac{\infty}{\infty}. \tag{32.5.2}$$

This is undefined! So we can't use the quotient law—at least in the way we've used it. We failed, like we've failed before. That's okay. We keep trying.

Here's a wonderful trick: Let's divide top and bottom of the function in question by x^3. Then we obtain:

$$\lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}{3\frac{x^3}{x^3}-3\frac{x^2}{x^3}}.$$

Let's follow this til the end:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+x+1}{3x^3-3x^2} &= \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}{3\frac{x^3}{x^3}-3\frac{x^2}{x^3}} && (32.5.3)\\
&= \frac{\lim_{x\to\infty} \frac{x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}{\lim_{x\to\infty} 3\frac{x^3}{x^3}-3\frac{x^2}{x^3}} && (32.5.4)\\
&= \frac{\lim_{x\to\infty} 1+\frac{1}{x^2}+\frac{1}{x^3}}{\lim_{x\to\infty} 3-3\frac{1}{x}} && (32.5.5)\\
&= \frac{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{1}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 3-\lim_{x\to\infty} 3\frac{1}{x}} && (32.5.6)\\
&= \frac{1+0+0}{3-3\cdot 0} && (32.5.7)\\
&= \frac{1}{3}. && (32.5.8)
\end{aligned}$$

The first line was the "divide top and bottom by x^3" trick, the next was the quotient rule, then we did some algebra. We obtain (32.5.6) using the addition rule, and then we obtain (32.5.6) by evaluating the limits we already knew how to evaluate. The final line is just arithmetic.

Here is the general trick: **When computing limits of rational functions *at* $\pm\infty$, divide the top and bottom by the highest power of x you see in the denominator.**

Example 32.5.2. Here is the work showing how to compute a few limits:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+x+1}{3x^3-3x^2} &= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{3-3\frac{1}{x}} && (32.5.9)\\
&= \frac{\lim_{x\to\infty} \frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{\lim_{x\to\infty} 3-3\frac{1}{x}} && (32.5.10)\\
&= \frac{0+0+0}{3-0} && (32.5.11)\\
&= 0. && (32.5.12)
\end{aligned}$$

$$\lim_{x\to\infty} \frac{x^4+x+1}{3x^2-3x} = \lim_{x\to\infty} \frac{x^2+\frac{1}{x}+\frac{1}{x^2}}{3-3\frac{1}{x}} \tag{32.5.13}$$

$$= \frac{\lim_{x\to\infty} x^2+\frac{1}{x}+\frac{1}{x^2}}{\lim_{x\to\infty} 3-3\frac{1}{x}} \tag{32.5.14}$$

$$= \frac{\lim_{x\to\infty} x^2+0+0}{3-0} \tag{32.5.15}$$

$$= \lim_{x\to\infty} \frac{x^2}{3} \tag{32.5.16}$$

$$= \frac{1}{3}\lim_{x\to\infty} x^2 \tag{32.5.17}$$

$$= \frac{1}{3}\cdot\infty \tag{32.5.18}$$

$$= \infty. \tag{32.5.19}$$

The reason this trick works: When you divide the denominator by the highest power of x you see there, you'll always end up with a denominator that looks like

$$\text{some number} + a\frac{1}{x} + b\frac{1}{x^2} + \ldots (\text{some coefficient})\frac{1}{x^k}.$$

But if we take the limit of this expression as $x \to \pm\infty$, we get the same "some number," because all other terms go to zero. In particular, the denominator is an actual number, so we'll never run into a quotient that's undefined.

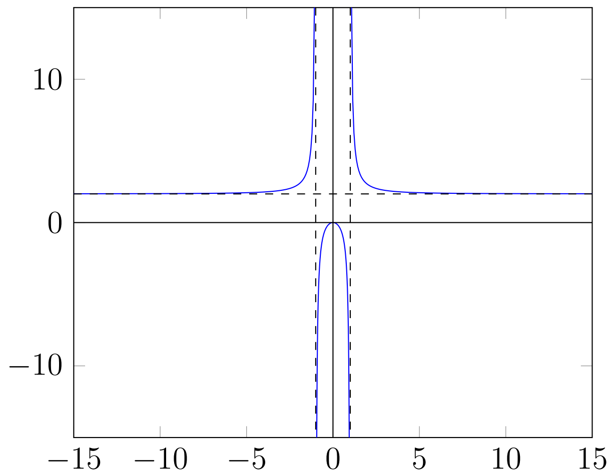
32.6 Asymptotes

For next lecture, I am going to have you practice finding horizontal and vertical asymptotes. The two things you'll be learning are (i) asymptotes, and (ii) computing limits using $0^\pm$ notation—this is a way to improve upon the quotient rule for limits.

First, asymptotes. Here is a vague definition:

Definition 32.6.1. As *asymptote* of a function is a line that approximates the function in some limit.

Example 32.6.2. Below is the graph of the function $f(x) = \frac{2x^2}{x^2-1}$:



I have drawn three dashed lines. Two of them are vertical, at $x = 1$ and $x = -1$. The other is horizontal, at height $y = 2$.

As you can see, as x approaches 1 or -1, the graph of the function begins to look more and more vertical, and the graph becomes near and nearer to the vertical lines . These two vertical lines are called *vertical asymptotes*. They are the lines $x = 1$ and $x = -1$.

You can also see that as x approaches ∞, the graph of f becomes closer and closer to the dashed horizontal line (of height 2). We say that the line $y = 2$ is a *horizontal asymptote* of f.

As it happens, f approaches the same line as x goes to $-\infty$. (This does not need to happen for $y = 2$ to be considered a horizontal asymptote; the graph might approach different horizontal asymptotes at ∞ and at $-\infty$.)

From the way I've described things, you've probably noted the following:

1. We find vertical asymptotes of f by seeing whether $\lim_{x\to a^+} f$ or $\lim_{x\to a^-}$ equals $\pm\infty$ at some a. If this limit does equal $\pm\infty$ at a, then the line $x = a$ is a vertical asymptote of f.

2. We find horizontal asymptotes of f by computing $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For example, if $\lim_{x\to\infty} f(x) = B$, then f has a horizontal asymptote of height B, because f approaches the horizontal line $y = B$ as x increases. And if $\lim_{x\to-\infty} f(x) = C$, then f also has a horizontal asymptote of height C, because f approaches the horizontal line $y = C$ as x approaches $-\infty$.

32.6.1 The 0^+ and 0^- notation (improving the quotient rule)

There is one more incredibly useful trick for knowing whether a limit is $\pm\infty$. Let me state the fact below as a Lemma; we'll improve upon it shortly:

Lemma 32.6.3. Suppose f is a quotient of two functions, so that

$$f(x) = \frac{g(x)}{h(x)}.$$

Suppose that

(i) $\lim_{x\to a^+} g(x)$ is positive [2], and

(ii) $h(x)$ approaches 0 from the right as x approaches a from the right.

Then

$$\lim_{x\to a^+} f(x) = \infty.$$

Example 32.6.4. Let us study $\lim_{x\to 3^+} \frac{1}{x-3}$. We see that as x approaches 3 from the right (meaning x is always larger than 3, but approaching 3), the expression $x-3$ is always positive, but is approaching 0. In other words, $x-3$ is approaching 0 from the right. Thus the lemma applies, and we conclude

$$\lim_{x\to 3^+} \frac{1}{x-3} = \infty.$$

(We already knew this fact, but we are formalizing it using the Lemma/trick above.)

Of course, if the denominator approaches 0 from the *left*, then the limit is no longer ∞, but is $-\infty$. So the lemma has different versions for how x approaches a, and how we approach 0 in the denominator. But more importantly, we are *very* lazy, and we don't want to have to say the words "the denominator approaches zero from the right/left" all the time.

So we will have a shorthand notation (Notation 32.6.5)—and it can cause confusion, so be careful. But because of this shorthand notation, the Lemma above will become compressed into some simple equalities, which you'll find in Lemma 32.6.8 below. It's these simple equalities that you'll *actually* be using when writing out work quickly.

[2] In particular, it is not equal to zero; but we do allow for this limit to be ∞

Notation 32.6.5 ($0^\pm$)**.** Let h be a function, and suppose that as x approaches a from the right, $h(x)$ approaches zero from the right. Then we will write

$$\lim_{x\to a^+} h(x) = 0^+.$$

Likewise, if $h(x)$ approaches zero from the left, we will write

$$\lim_{x\to a^+} h(x) = 0^-.$$

We use the same notation when we have x approach a from the left, too. So we can write things like

$$\lim_{x\to a^-} h(x) = 0^+ \qquad \text{or} \qquad \lim_{x\to a^-} h(x) = 0^-.$$

Warning 32.6.6. Unlike $\pm\infty$, I will **discourage** you from thinking of 0^+ and 0^- as numbers. You should think of 0^+ as shorthand for "approaching zero from the right," and the equality symbol of $\lim_{x\to a^-} h(x) = 0^+$ *not* as an equality of numbers, but a shorthand for saying "0^+ is the way that this limit looks."

Ah, but there are always caveats. See the footnote below.[3]

Example 32.6.7. The following are all correct uses of this notation:

1. $\lim_{x\to 3^+} \frac{1}{x-3} = \frac{1}{0^+}$.
2. $\lim_{x\to 3^-} \frac{1}{x-3} = \frac{1}{0^-}$.
3. $\lim_{x\to 3^+} \frac{1}{3-x} = \frac{1}{0^-}$.
4. $\lim_{x\to 3^-} \frac{1}{3-x} = \frac{1}{0^+}$.

Here is the condensed version of Lemma 32.6.3 above, and of its relatives:

Lemma 32.6.8. Let A be positive. (A can be a number, or it can equal ∞.) Then

$$\frac{A}{0^+} = \infty \qquad \text{and} \qquad \frac{A}{0^-} = -\infty.$$

[3]There *is* a system of numbers for which you can think of both $\pm\infty$ and $0^\pm$ as legitimate "numbers." But this can be a little confusing at first glance, and a discussion about this can take us very, very far astray, so we won't be exploring this avenue in this class. But I hope you see that a door is cracked open: A door to a place where you can explore new notions of "number" and test your imagination against mathematical truths.

If instead A is negative, then

$$\frac{A}{0^+} = -\infty, \qquad \text{and} \qquad \frac{A}{0^-} = \infty.$$

(End of Lemma.)

Warning 32.6.9. Expressions like $\frac{A}{0^+}$ *only arise when computing one-sided limits.* I warn you again to only use the above lemma when you are computing certain one-sided limits, and to not think of the equalities above as equalities of numbers, but a shorthand, lazy way of saying (for example) "If the denominator approaches 0 from the right, and if A is positive, then the limit is ∞."

Remark 32.6.10 (A helpful way to think about Lemma 32.6.8)**.** Usually, we can't compute limits when the denominator equals zero. Think of Lemma 32.6.8 as a way of *improving* the quotient rule: So long as we are computing one-sided limits, and so long as we do not get a result that looks like "0/0," we can actually compute limits even when the denominator approaches zero (so long as the denominator only approaches zero from one side).

Example 32.6.11. Here are examples of how you can use the above notation to write out the work to compute some limits:

(i)

$$\lim_{x\to 3^+} \frac{x}{x-3} = \frac{\lim_{x\to 3^+} x}{\lim_{x\to 3^+} x - 3} \tag{32.6.1}$$

$$= \frac{3}{\lim_{x\to 3^+} x - 3} \tag{32.6.2}$$

$$= \frac{3}{0^+} \tag{32.6.3}$$

$$= \infty. \tag{32.6.4}$$

I used the "dividing by 0^+" notation from Lemma 32.6.8 in the last two equalities. Everything else is a straightforward application of limit laws.

(ii)

$$\lim_{x\to 3^-} \frac{x}{x-3} = \frac{\lim_{x\to 3^-} x}{\lim_{x\to 3^-} x - 3} \tag{32.6.5}$$

$$= \frac{3}{\lim_{x\to 3^-} x - 3} \tag{32.6.6}$$

$$= \frac{3}{0^-} \tag{32.6.7}$$

$$= -\infty. \tag{32.6.8}$$

I used the "dividing by 0^+" notation from Lemma 32.6.8 in the last two equalities. Everything else is a straightforward application of limit laws.

(iii)

$$\lim_{x\to 3^-}\frac{x^2}{9-x^2} = \frac{\lim_{x\to 3^-} x^2}{\lim_{x\to 3^-} 9-x^2} \tag{32.6.9}$$
$$= \frac{9}{\lim_{x\to 3^-} 9-x^2} \tag{32.6.10}$$
$$= \frac{9}{0^+} \tag{32.6.11}$$
$$= \infty. \tag{32.6.12}$$

The important thing to note here is the step from (32.6.10) to (32.6.11). Though x is approaching 3 from the left, $9-x^2$ is approaching zero *from the right*; this is because as x approaches 3 from the left, x^2 is always less than 9; so $9-x^2$ is always positive. In contrast, we have:

$$\lim_{x\to 3^+}\frac{x^2}{9-x^2} = \frac{\lim_{x\to 3^+} x^2}{\lim_{x\to 3^+} 9-x^2} \tag{32.6.13}$$
$$= \frac{9}{\lim_{x\to 3^+} 9-x^2} \tag{32.6.14}$$
$$= \frac{9}{0^-} \tag{32.6.15}$$
$$= -\infty. \tag{32.6.16}$$

Putting things together Now we can put the two new ideas of this packet together.

Example 32.6.12. Find all vertical and horizontal asymptotes (if any) of

$$f(x) = \frac{4x^3 + 3x - 2}{3x^2 - 27}$$

(i) Let's first begin to look for horizontal asymptotes. Remember, this means looking for limits as x approaches $\pm\infty$. And the trick for this for rational functions is to divide the numerator and denominator by the highest power in the denominator. So let's begin our computation doing that, and go on:

$$\lim_{x\to\infty} \frac{4x^3 + 3x - 2}{3x^2 - 27} = \lim_{x\to\infty} \frac{4x^3 + 3x - 2}{3x^2 - 27} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \tag{32.6.17}$$

$$= \lim_{x\to\infty} \frac{4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{3 - \frac{27}{x^2}} \tag{32.6.18}$$

$$= \frac{\lim_{x\to\infty} 4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{\lim_{x\to\infty} 3 - \frac{27}{x^2}} \tag{32.6.19}$$

$$= \frac{\lim_{x\to\infty} 4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{3 - 0} \tag{32.6.20}$$

$$= \frac{\lim_{x\to\infty} 4x + 3\frac{1}{x} - 2\frac{1}{x^2}}{3} \tag{32.6.21}$$

$$= \frac{\lim_{x\to\infty} 4x + \lim_{x\to\infty} 3\frac{1}{x} - \lim_{x\to\infty} 2\frac{1}{x^2}}{3} \tag{32.6.22}$$

$$= \frac{\infty + 0 - 0}{3} \tag{32.6.23}$$

$$= \infty. \tag{32.6.24}$$

Because this limit is not a real number, there is *no* horizontal asymptote that f approaches as x goes to ∞.

An almost identical computation will show that $\lim_{x\to-\infty} f(x) = -\infty$, so that there is no horizontal asymptote that f approaches as x goes to $-\infty$, either. In sum, there are no horizontal asymptotes.

(ii) Finally, let's check for vertical asymptotes. This has to do with checking when the denominator might limit to zero. So we must find when the expression

$$3x^2 - 27$$

could equal zero. This happens when $x^2 = 9$, meaning we must study the limits as x approahces ± 3. All we need to check is, for each of these values, whether *either* of the one-sided limits approaches $\pm\infty$.

So we must compute

$$\lim_{x\to 3^+} \frac{4x^3+3x-2}{3x^2-27}. \tag{32.6.25}$$

The numerator becomes

$$4(3)^3 + 3\cdot 3 - 2 = 4\cdot 27 - 9 - 2 = 108 - 9 - 2 = 97.$$

On the other hand, the denominator approaches 0 from the right as x approaches 3 from the right. So we have

$$\lim_{x\to 3^+} \frac{4x^3+3x-2}{3x^2-27} = \frac{97}{0^+} = \infty.$$

So we have found a vertical asymptote at $x = 3$. (At this point, we are happy with $x = 3$, and we don't need to check the lefthand limit at 3.)

Let's make sure that we have a vertical asymptote at $x = -3$. For example, you will find

$$\lim_{x\to -3^+} \frac{4x^3+3x-2}{3x^2-27} = \frac{\text{some non-zero number}}{0^-} = -\infty,$$

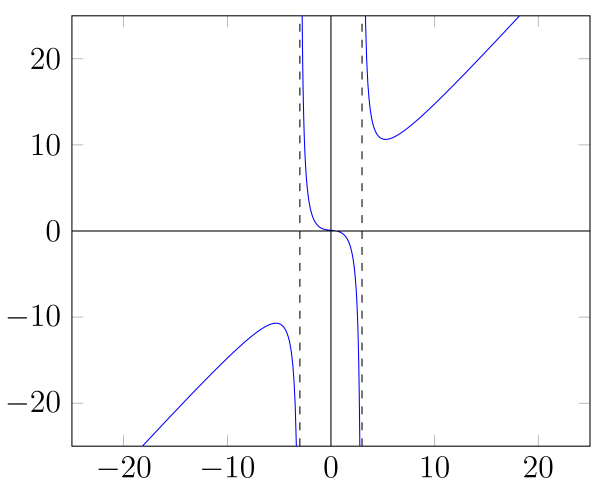
and

$$\lim_{x\to -3^-} \frac{4x^3+3x-2}{3x^2-27} = \frac{\text{some non-zero number}}{0^+} = \infty.$$

Computing either of these one-sided limits shows that there is a vertical asymptote at $x = -3$.

To summarize: $f(x)$ **has no horizontal asymptotes, but has two vertical asymptotes at $x = 3$ and $x = -3$.**

In case you want to check your answer, here is a graph of the function:



32.7 For next time

For next lecture, I expect you to be able to find the vertical and horizontal asymptotes for the following functions:

(a) $f(x) = \frac{1}{x-4}$

(b) $f(x) = \frac{x^2}{x^2-9}$

(c) $f(x) = \frac{x^3}{x^2-9}$