Lecture 27

Epsilon-Delta

(Make sure you look over the notes on the composition law from last class.)

27.1 Motivating example

Suppose you have a machine which takes x grams of serum and outputs $f(x)$ grams of pure vaccine. The machine is very accurate and reliable.

However, your measurement of serum is sometimes imprecise. For example, it's just a fact of life that we sometimes can't measure out exactly 5 grams of serum, and that we actually end up putting in 5.0001 grams of serum instead. There is always some uncertainty or error. As a result, there is always some uncertainty in how much vaccine we will output.

Let's say you want to produce about L grams of useful vaccine. Let's say you suspect you need a grams of serum to produce L grams of vaccine because you believe $f(a) = L$. As mentioned, it may be hard to measure out the precise amount of serum you need, so *guaranteeing* a product of exactly L grams of vaccine is hard. So let's relax the goal a little bit: Let's choose some small acceptable "error number" ϵ, and let's try to produce at least $L - \epsilon$ grams.

(Here, ϵ is the Greek letter "epsilon." It later evolved into the letter e; and you can think of it as standing for "error.")

And certainly, you don't want to produce more vaccine than you need to, so let's try to produce at most $L + \epsilon$ grams of vaccine. That is, instead of trying to produce exactly L grams of vaccine, we aim to produce somewhere between $L + \epsilon$ and $L - \epsilon$ grams of useful vaccine. If we want to produce this much vaccine, then we might not need to put in exactly a grams of serum; but we can instead put in x grams of serum for any value x "close enough" to a.

Put another way, let's try to put in a "correct enough" amount of serum, x, so that $f(x)$ is between $L-\epsilon$ and $L+\epsilon$. An inequality expressing this desired x is

$$|f(x) - L| < \epsilon$$

(which states that $f(x)$ and L are at most ϵ away).

You suspect that if you put in exactly a grams of serum, you'll get out exactly L. So if your x is a little bit inaccurate, you suspect that $f(x)$ will also be only a little off of L. Hmm. How accurate do you need the *input* to be to make sure you produce the desired amount of vaccine?

Input accuracy is how close x is to your desired a. In other words, accuracy can be measured by the number $|x-a|$. To be "accurate enough" means that you want the deviation of x from a—also known as $|x-a|$—to be small enough to guarantee that $|f(x) - L| < \epsilon$.

Here's what we expect, then:

Tell me how small you want the ϵ error to be. If we know that $|x-a|$ is small enough, then we can guarantee that $|f(x) - L| < \epsilon$.

To be as mathy as we can, let's also give a mathy expression for $|x-a|$ to be "small enough." To do this, let's choose some number δ, and let's demand that $|x-a| < \delta$.[1] Then we know that x and a cannot be δ or more apart.

Tell me how small you want the ϵ error to be. Then I can tell you a δ so that, whenever $|x-a|$ is less than δ, we can guarantee that $|f(x) - L|$ is less than ϵ.

So δ is a measure of how accurate your input needs to be to guarantee that your output is within the acceptable error ϵ.

In sum: Given the error tolerance ϵ, you want to find the permitted deviation δ to produce an amount within the tolerable range.

Remark 27.1.1. δ is the measure of your needed accuracy. If a grams of serum outputs exactly L grams of vaccine, a tiny mistake in measuring x grams—that is, a small enough mistake that you actually put in somewhere between $a-\delta$ and $a+\delta$ grams—should guarantee that you output between $L-\epsilon$ and $L+\epsilon$ grams of useful vaccine. You just need to know how small "small enough" actually is! That is, how small does δ need to be once you're aiming for ϵ error?

27.2 Limits defined

Here is a definition that is notoriously difficult for calculus students:

[1] δ is the Greek letter lower-case 'delta.' It is the old form of the letter d. You can think of δ as standing for the allowed "deviation" of the variable.

Definition 27.2.1. Let $f(x)$ be a function, and choose a number a.

We say that $f(x)$ *has a limit at a* if the following holds:

There exists a number L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ for which

$$\text{if } x \neq a \text{ and } |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

If $f(x)$ has a limit at a, we call L the limit of $f(x)$ at a.

This can be a very confusing definition. But our previous example captures exactly the intuition! Let's break down the definition bit by bit:

1. "for every ϵ" should be thought of as *for every acceptable output error*.

2. "There exists a δ" should be thought of as "You can find some acceptable input deviation"

3. now, the "$x \neq a$" is a bit technical so you can ignore; but it has to do with the fact that we'll want to apply the puncture law. The value $f(a)$ doesn't actually matter to compute the limit at a, so the "$x \neq a$" condition helps us throw out the unnecessary data of $f(a)$.

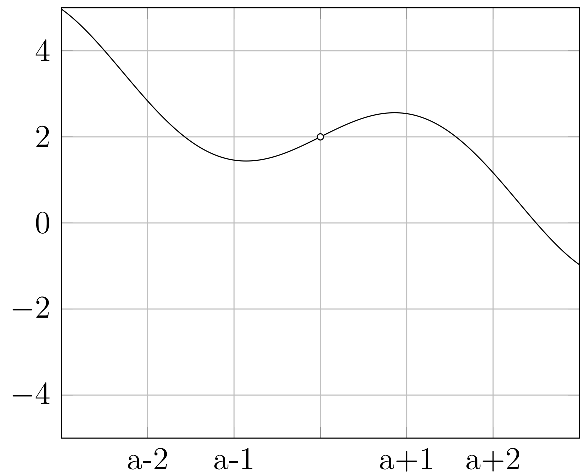
4. But "if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$" is exactly the accuracy statement we want: So long as x does not deviate from a more than δ, then $f(x)$ will not deviate from L more than ϵ.

5. Finally, this expected goal "L" is the limit. In other words, it is supposed to fit our intuition of "the value that $f(x)$ approaches as x approaches a." This tells you how we mathematically think of approaching something like L: You say how close you want to be to L, and you can guarantee that you'll be that close to L so long as you're close enough to a.

The following gives some further pictures and ideas to help you think about what's going on.

27.3 Exploring ϵ-δ visually

Below is the graph of a function $f(x)$, undefined at $x = a$.



Based on the graph, we suspect that

$$\lim_{x \to a} f(x) = 2.$$

Exploratory questions.

(i) Can we guarantee that so long as x is close enough to a, then $f(x)$ is within 0.4 units of the suspected limit?

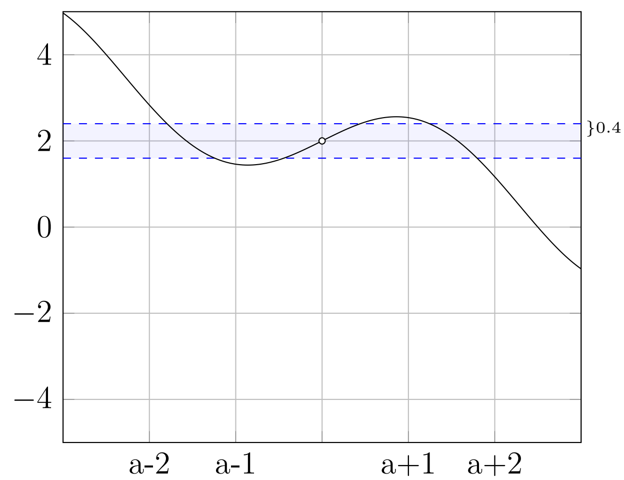
(ii) If so, how close does x have to be to a?

(a) On the graph above, draw the region of all points on the plane whose vertical coordinate is *strictly* between $2 - 0.4$ and $2 + 0.4$. (That is, between 1.6 and 2.4, non-inclusive.) Your answer should look like a horizontal strip.

(b) Does drawing this strip help you visualize the main questions?

(c) As we will see, the take-away here is that you *can* guarantee to be within 0.4 of the suspected limit, so long as you choose x to be close enough to a.

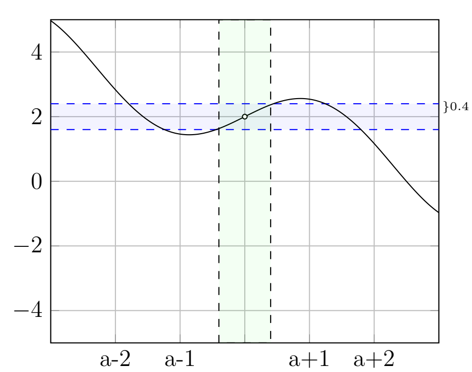
Recap

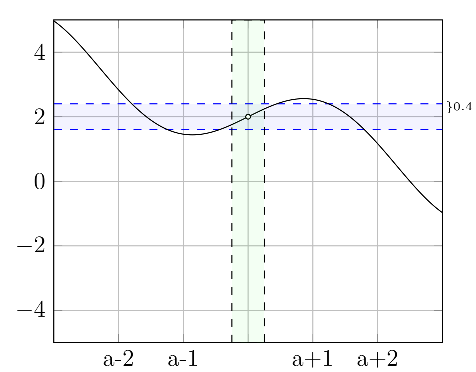


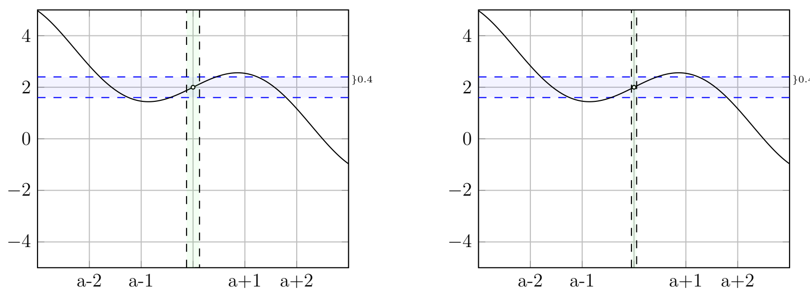
(a) Above, we have drawn the solution to (a) of the previous page. It is the strip between the line of height 2+0.4, and the line of height 2-0.4. Note that the edges of the strips are dashed, so that the vertical coordinates of the points in the strip are strictly between 1.6 and 2.4 (and not equal to either value).

(b) This helps us answer the main questions: So long as the graph of $f(x)$ is inside the strip, we know that $f(x)$ is within 0.4 of the suspected limit! (Remember that the suspected limit is 2.)

(c) Now, visually, we notice that in a region where x is close enough to a, the graph of $f(x)$ is always inside the strip. Here are sample examples:

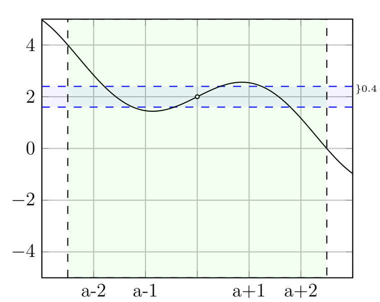




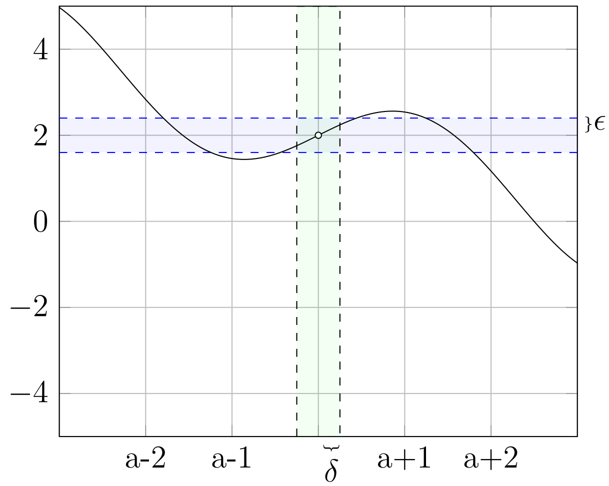


That is, *so long as x is in a thin-enough vertical strip, the graph of q in that vertical strip will also be in the horizontal strip.*

Warning. The "thin-enough" is important. If the vertical strip is too wide (that is, if we allow x to take values that are too far away from a) then $f(x)$ may escape the horizontal strip, meaning the value of $f(x)$ may be more than 0.4 away from the suspected limit. Below is an example where, because the vertical strip is too wide, the portion of $f(x)$ within the vertical strip is *not* contained in the horizontal strip.



Reading this will save you a lot of trouble. In ϵ-δ proofs, the vertical strips are always of width 2δ. The horizontal strips are always of height 2ϵ.



27.3.1 The ϵ-δ definition, in detail

Our mission is to understand the following statement:

$$\text{“}L \text{ is the limit of } f(x) \text{ as } x \text{ goes to } a.\text{”} \tag{27.3.1}$$

Informally, the above statement—(27.3.1)—can be rephrased as follows:

$$\text{“So long as } x \text{ is close enough to } a, \text{ we know } f(x) \text{ is close to } L.\text{”} \tag{27.3.2}$$

Now, we are going to interpret x being "close to a" as follows: that x is contained in some thin, vertical strip centered at a.

Likewise, we will interpret "$f(x)$ is close to L," as "$f(x)$ is contained in some thin, horizontal strip centered at L."

Let me stress again that the vertical strip is centered at a, and the horizontal strip is centered at height L.

In the drawings on the previous pages, we saw we could pictorially retranslate (27.3.2) to the following:

"So long as the x-coordinate is contained in some thin-enough vertical strip, we know $f(x)$ is contained in some thin horizontal strip." (27.3.3)

Now we will leap from the word "strip" to some algebraic notation. If we say that the vertical strip has width 2δ, then to say that the x-coordinate is contained in the vertical strip of 2δ centered at a is to say that

$$|x - a| < \delta.$$

Read that again if you didn't get it.

Likewise, to say that $f(x)$ is contained in a horizontal strip of height 2ϵ, centered at L, is to say that

$$|f(x) - L| < \epsilon.$$

Make sure you understand these inequalities.

Then, the statement (27.3.3) can finally be re-written as follows:

$$\text{"So long as } |x - a| < \delta, \text{ we know } |f(x) - L| < \epsilon.\text{"} \tag{27.3.4}$$

Make sure you understand how we got from (27.3.3) to (27.3.4).

So we see how ϵ, δ, and those confusing-looking inequalities show up.

But our definition also has a condition about $x \neq a$—this is just to emphasize that the limit a doesn't depend on the value of f *at* a, it only depends on the values of f at points *close to* a.

Let me put the cherry on top. The ϵ-δ definition of limit is equivalent to asserting the following: If $\lim_{x \to a} f(x) = L$, then you can always win a game.

What game? Your enemy dares you to fit the graph of $f(x)$ inside some strip of height 2ϵ. The only clue you are given is *which* ϵ your enemy chooses. You win if you can find a *width*, which we will call 2δ, so that whenever x is inside the vertical strip of that width, you know that the graph of $f(x)$ is within the horizontal strip with enemy-specified height.

27.3.2

Now, when you can find a δ given any ϵ, your machine is a great one. Mathematically, this greatness translates to "$f(x)$ has a limit."

But you might have a machine that is completely unpredictable and unreliable for certain amounts of input serum. It just gets downright finicky when $a = 10$. And for some values of ϵ, no matter how small you can reduce your inaccuracy δ, you just cannot guarantee an output within the error tolerance of ϵ. That's an unfortunately bad machine. This mathematically translates into "$f(x)$ does not have a limit at 10."

27.4 Playing with ϵ-δ algebraically

For the next few days, we will explore something called ϵ-δ proofs (this is read as "epsilon-delta" proofs).

Here is the general principle: Given a function g, a suspected limit L for g at a, and an error number ϵ, you must find a δ (read *delta*) that guarantees you can get within ϵ (*epsilon*) of the suspected limit after applying g.

Example 27.4.1. Let $g(x) = (8x^2 + x)/x$. You suspect that the limit of $g(x)$ as x approaches zero is 1. (You might arise at such a suspicion by simplifying g, or drawing a graph of g.) So in this example, $L = 1$.

Now, for no good reason, let's say somebody says they want to limit the error to $\epsilon = 0.1$. Can you find a positive number δ so that, so long as you choose a $x \neq 0$ with $|x| < \delta$, then $f(x)$ is within ϵ of 1? (Put another way, so long as x is small enough—meaning its absolute value is less than δ—then the value of $g(x)$ is very close to 1—meaning at most distance ϵ from 1.)

Yes, you can.

To see how you can find this δ, let's note the following:

$$|g(x)-L| = |g(x)-1| = |\frac{8x^2+x}{x} - \frac{x}{x}| = |\frac{8x^2+x-x}{x}| = |\frac{8x^2}{x}| = |8x| \qquad (\text{when } x \neq 0)$$

The very lefthand side of this expression is the distance between $g(x)$ and the suspected limit, 1. The very righthand side is telling you that this distance is always given by $|8x|$ when $x \neq 0$. So for example, if you took x to be 0.2, then the distance between $g(x)$ and your suspected limit would be $|8x| = |8 \times 0.2| = 1.6$.

So if you want $g(x)$ to be within ϵ of 1, you want $|8x|$ to be less than ϵ. That is, you want

$$|8x| < \epsilon.$$

This happens so long as $|x| < \epsilon/8$. So, choose $\delta = \epsilon/8$. Then so long as $|x| < \delta$, you can guarantee that $|g(x) - 1| < \epsilon$.

Note that while I originally asked for a δ so that you are within 0.1 of the suspected limit, you have discovered that regardless of ϵ, you can choose $\delta = \epsilon/8$ to be within ϵ of the suspected limit. If $\epsilon = 0.1$ as in our original problem, we can tell people to choose $\delta = 0.1/8$ to guarantee that the error is less than 0.1.

In fact, you can tell people to choose a deviation δ that is even *less* than 0.1/8 if you want! After all, the smaller your deviation, the smaller your error. So there isn't a "single" answer for δ. So long as you can convince me that your deviation δ is small enough to guarantee a small error, your answer is correct.

Example 27.4.2. Let $g(x) = 3x/x$. You suspect that the limit of $f(x)$ as x approaches zero is 3.

And let $\epsilon = 12$. Can you find a positive number δ so that, so long as $x \neq 0$ and $|x| < \delta$, then $g(x)$ is within ϵ of 3?

Yes; in fact, any positive number δ will do. This is because—regardless of x—$g(x)$ is always equal to 3 so long as $x \neq 0$. Thus

$$|g(x) - 3| = |\frac{3x}{x} - 3| = 0 \qquad \text{whenever } x \neq 0$$

and 0 is of course smaller than any ϵ. So, regardless of δ, your $g(x)$ will always be within δ of 3.

Example 27.4.3. Let $g(x) = (4x^3 + 9x)/x$. You suspect that the limit of $g(x)$ as x approaches zero is 9. (You might arise at such a suspicion by simplifying g, or drawing a graph of g.)

Now suppose someone gives you some positive number called ϵ. Can you find a positive number δ so that, so long as you choose a value of x so that $x \neq 0$ and $|x| < \delta$, then $g(x)$ is within ϵ of 9? That is, can you find a δ so that

$$|x| < \delta, x \neq 0 \qquad \text{implies} \qquad |g(x) - 9| < \epsilon?$$

Yes, you can.

To see how you can find this δ, let's note the following:

$$|g(x) - 9| = |\frac{(4x^3 + 9x)}{x} - \frac{9x}{x}| = |\frac{4x^3 + 9x - 9x}{x}| = |\frac{4x^3}{x}| = |4x^2| \qquad (\text{when } x \neq 0)$$

The very lefthand side of this expression is the distance between $g(x)$ and the suspected limit, 9. The very righthand side is telling you that this distance is always given by $|4x^2|$ when $x \neq 0$. So for example, if you took x to be 0.1, then the distance between $g(x)$ and your suspected limit would be $|4x^2| = |4 \times 0.01| = 0.04$.

So if you want $g(x)$ to be within ϵ of 9, you want $|4x^2|$ to be less than ϵ. That is, you want

$$|4x^2| < \epsilon.$$

That is, you want

$$|x^2| < \epsilon/4.$$

Because squaring a number preserves $<$—meaning $a^2 < b^2$ if and only if $|a| < |b|$—we conclude that for the above inequality to hold, we want

$$|x| < \sqrt{\epsilon/4}.$$

Thus, set $\delta = \sqrt{\epsilon/4}$. Then, based on the work above, we know that if $|x| < \delta$, then $|g(x) - 9|$ is less than ϵ.

27.5 For next time

For next quiz, you will be tested on whether—given $g(x)$, a suspected limit L, and ϵ—you can find a δ so that

$$\text{If } x \neq 0 \text{ and } |x| < \delta, \text{ then } |g(x) - L| < \epsilon.$$

You will be quizzed on the following g and L. (You should be able to find δ as an expression involving only g, L, ϵ, though often you will not need L at all.)

1. $g(x) = (2x^3 + 9x)/x$, with $L = 9$.
2. $g(x) = (5x^2 + 7x)/x$, with $L = 7$.
3. $g(x) = 3$, with $L = 3$.