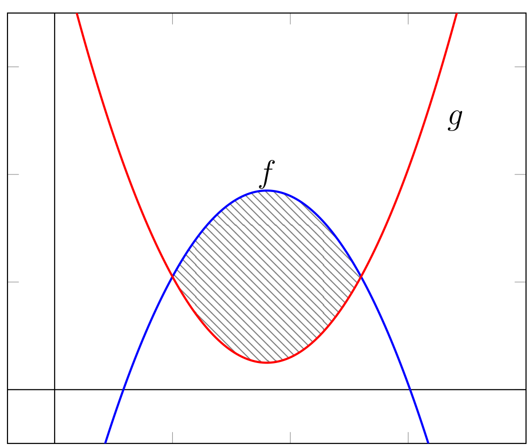
Lecture 21

Areas between curves

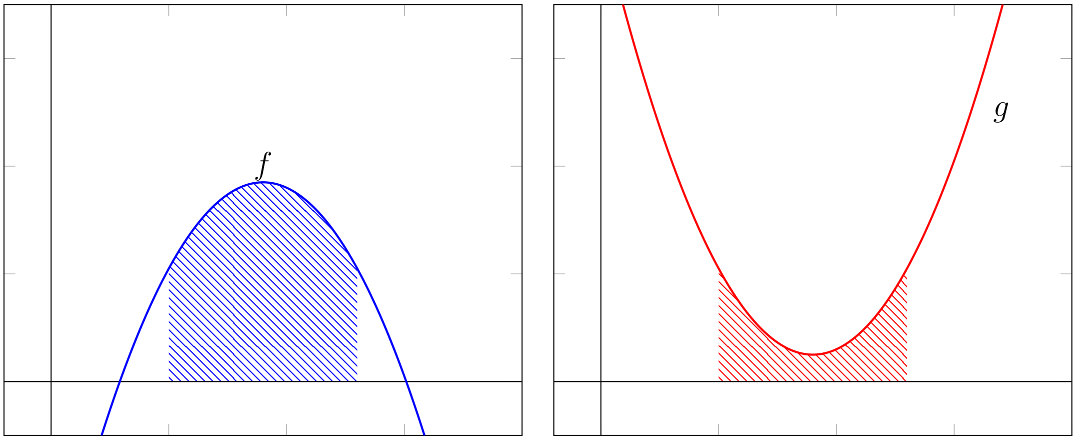
We've so far seen how to find areas of regions between (graphs of) functions and the x-axis. But we can make more interesting shapes by looking at regions between graphs of *two* functions.

Below is a picture of two functions, f and g. f is in blue, and is concave down. g is in red, and is concave up.

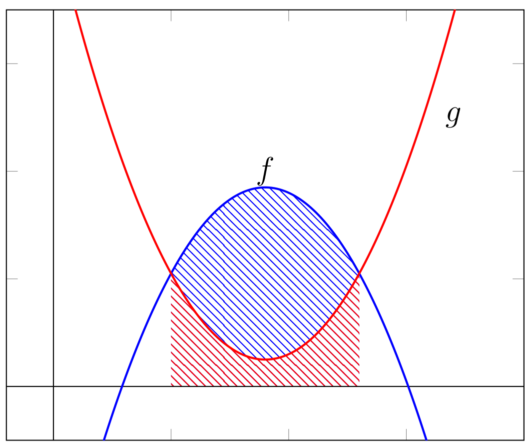
How would we find the area of the shaded region?



Well, let's look at this region as obtained by taking a big region, and subtracting off another. Observe:



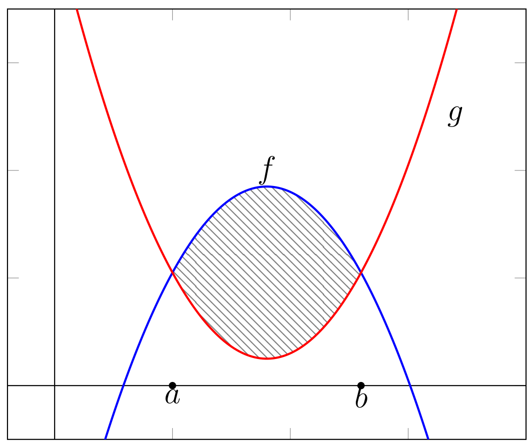
In blue is the area between the graph of f and the x-axis, while in red is the area between the graph of g and the x-axis. Overlaying the pictures, we see that our original region is obtained by removing the red region from the blue region.



So in this particular example, we can conclude that

$$\text{Area of region between } f \text{ and } g = \int_a^b f\,dx - \int_a^b g\,dx.$$

Here, a and b are where the graphs of f and g intersect; they are the rightmost and leftmost points of the region we seek:



The upshot is as follows:

Proposition 21.0.1. Suppose that the graphs of $f(x)$ and $g(x)$ intersect at the points a and b, with $a < b$. Suppose also that $f(x) \geq g(x)$ for all points x along the interval $[a, b]$. Then the area of the region formed by $f(x)$ and $g(x)$ is given by

$$\int_a^b f(x) - g(x)\, dx.$$

For a problem like this, you may typically be given values of a and b, or you may have to find the values of a and b yourself. To find a and b, you have to solve for the numbers a and b at which $f(a) = g(a)$ and $f(b) = g(b)$.

Example 21.0.2. Find the area between the graphs of the functions $\cos(x)$ and $x^2 - 1 + \cos(x)$.

We must identify where the two functions intersect. This happens when

$$\cos(x) = x^2 - 1 + \cos(x).$$

Solving this equation, we arrive at the conclusion that x must equal -1 or 1. Since $-1 < 1$, we set $a = -1$ and $b = 1$.

Next we must decide which function is larger than (i.e., on top of) the other. We can test this at any point between -1 and 1, so let's try $x = 0$. Then $(0^2) - 1 + \cos(0)$ is less than $\cos(0)$, so we let $\cos(0)$ be the "on top" function. The proposition above tells us to subtract the bottom function from the top function, and integrate form a to b:

$$\int_{-1}^1 (\cos(x)) - \left(x^2 - 1 + \cos(x)\right)\, dx.$$

We can simplify the integrand before integrating:

$$= \int_{-1}^{1} -x^2 + 1\, dx.$$

Now we conclude

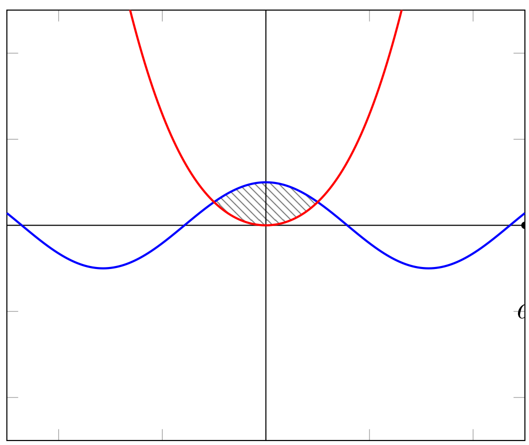
$$\int_{-1}^{1} -x^2 + 1\, dx = \frac{-1}{3}x^3 + x\Big|_{-1}^{1} \tag{21.0.1}$$

$$= \left(\frac{-1}{3}(1)^3 + (1)\right) - \left(\frac{-1}{3}(-1)^3 + (-1)\right) \tag{21.0.2}$$

$$= \left(\frac{2}{3}\right) - \left(\frac{-4}{3}\right) \tag{21.0.3}$$

$$= 2. \tag{21.0.4}$$

In case you are curious, here is what the region looks like:



21.1 Practice problems

Exercise 21.1.1. Determine the area of the region bounded by $y = \frac{8}{x}$, $y = 3x$ and $x = 5$.

Exercise 21.1.2. Determine the area of the region bounded by $x = 2+y^2$, $x = 1-y^2$, $y = 2$ and $y = -3$. (You may want to ask Hiro about this one.)

Exercise 21.1.3. Find the area of the region bounded by $y = x^2 - x - 6$ and $y = 2x + 4$.

Exercise 21.1.4. Find the area of the region bounded by $y = x + 1$ and $y = 9 - x^2$.

Exercise 21.1.5. Find the area of the region bounded by $y = x$ and $y = x^2$.

Exercise 21.1.6. Find the area of the region bounded by $x = 1 - y^2$ and $y = y^2 - 1$.

21.2 For next time

Keep practicing integration, and u substitution, and finding areas between curves.