Lecture 15

Practice problems for the midterm

15.1 Taking derivatives

Find the slope of the tangent line to the following functions at $x = 2$. You may leave your answers using symbols such as sin, ln, et cetera, but you should perform well-known simplifications (like $\sin(0) = 0$) for full credit. Approximate answers will not be given full credit; answers must be exact. (For example, 3.1415926 is *not* exact an exact value of π.)

(a) $f(x) = 3$

(b) $f(x) = 9x^3 + 7x$

(c) $f(x) = \sin(\pi x) - \ln(x)$

(d) $f(x) = \arcsin(x/3) - \arctan(x/3)$.

(e) $f(x) = \tan(x)$.

15.2 Taking derivatives, II

Find the slope of the tangent line to the following functions at $x = 2$. Same instructions as above.

(a) $f(x) = \sin(e^x)$.

(b) $f(x) = e^{\sin(x)}$.

(c) $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

(d) $f(x) = (e^{\cos(x)})^3$.

(e) $f(x) = e^{\ln(x)}$.

15.3 Word problems

Exercise 15.3.1. There is a comet in our solar system. The distance of the comet from the surface of earth is given by a function $d(t)$, where t is measured in days from today, and d is measured in astronomical units (one astronomical unit is approximately 93 million miles, though that won't matter for this problem). Your astronomers tell you that an accurate model for $d(t)$ is

$$d(t) = e^{((t-3)^2)}.$$

In terms of astronomical units, what is the closest that the comet will get to the earth? And how many days do we have until this closest approach?

Exercise 15.3.2. Let $f(t) = x^3 - 27$. Find all values of t at which f attains a local maximum or a local minimum.

Exercise 15.3.3. Consider the set of points (x, y) satisfying the equation

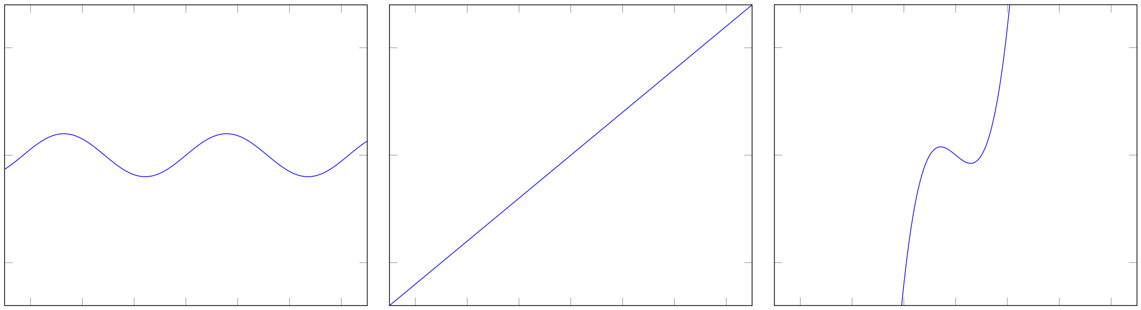
$$xy - \sin(y + 3x) = 1.$$

(You can graph this shape using an online grapher if you like; it's wonky! However, you answer must be based on calculus, not based on what you see online.)

(i) Given a point (x, y) on this shape, find the slope of the tangent line to that point. (Your answer should be in terms of x and y.)

(ii) At any point on this graph that touches the x-axis, what can you say about the slope of the tangent line to that point? (Hint: If $\sin\theta = 1$, what do you know about $\cos\theta$?)

Exercise 15.3.4. Below are the graphs of various functions f. Shade in all parts of the graphs along which f'' is positive.



Exercise 15.3.5. A version of the ideal gas law states that, all else being equal, the pressure, volume, and temperature of an ideal gas satisfy the following equation:

$$PV = kT$$

where k is some constant number. (It does not change with time.) Let t denote time in seconds—not to be confused with T, which represents temperature and is a function of time.

Suppose that we are told the value of k, and that we are conducting an experiment with an ideal gas where at $t = 3$ seconds,

1. The pressure is 5 Pascals.
2. The pressure is changing at 0.1 Pascals per second,
3. and the volume is constant.

Is this enough information to determine how quickly the temperature of the gas is changing at $t = 3$ seconds?

If not, what other information do you need?

Exercise 15.3.6. A star is born (in outer space). At time t years, the star has a sphere of radius $R(t)$, measured in kilometers.

If the star has a radius of 100,000 kilometers, and is growing in volume at 1,000,000 kilometers-cubed per year, how quickly is its radius growing?

For this problem, you will need to use that the volume of a sphere is given by $\frac{4}{3}\pi R^3$, where R is the radius of the sphere.

15.4 Challenge problems

Exercise 15.4.1. Using the Mean Value Theorem, prove that if a function $f(x)$ has zero derivative everywhere, then $f(x)$ must be constant. (Hint: Prove the contrapositive.)

Exercise 15.4.2. Construct a polynomial $T(x)$ of degree 5 whose nth derivative at $x = 0$ equals the nth derivative of $\sin(x)$ at $x = 0$ for $n = 0, 1, ..., 5$. (The "0th derivative" is just the value of the function.)

Try graphing $T(x)$ and $\sin(x)$. What can you see? Based on this, is there something you can compare about $T(0.1)$ and $\sin(0.1)$, for example?

(We will see more of this toward the end of the semester.)

Exercise 15.4.3. Let $f(x) = |x|$ be the absolute value function. Explain, using the difference quotient and our informal understanding of limits, why f does not have a derivative at $x = 0$. Explain why f does have a derivative for any other value of x.