Lecture 12

Mean Value Theorem, I

12.1 If-then statements, and converses

Last time we saw the following:

Proposition 12.1.1. Let x be a local extremum of f. Then x is a critical point of f.

The above proposition is using some new vocabulary from last time. remember, a *local extremum* is a point that is either a local minimum or maximum. (This means that the value of f at x is smaller than all nearby points' values, or larger than all nearby points' values.) And a critical point of f is a point x at which $f'(x) = 0$.

Another way to state the proposition is by using an if-then construction:

If *x is a local extremum of f*, then $\underline{x \text{ is a critical point of } f.}$

The italicized part is the *hypothesis* of the if-then statement, and the underlined part is the *conclusion* of the if-then statement.

I then asked in class whether if x is a critical point, then x must be a local extremum. We saw, in the example of $f(x) = x^3$, that this isn't true. For this function, $x = 0$ is a critical point, but $x = 0$ is not a local extremum. (By moving even a tiny bit to the right, f becomes larger, while moving even a tiny bit to the left, f becomes smaller.) In other words, the statement

"If *x is a critical point of f* then $\underline{x \text{ is a local extremum of } f}$"

is a **false** statement.

The two if-then statements above are very closely related. The two statements look identical, except the *hypothesis* and the conclusion have been swapped! We say that the two statements are **converses** to each other. (So the first statement is the converse of the second statement, and the second statement is the converse of the first statement.)

Remark 12.1.2 (Converse of, converse to)**.** The prepositions involved can be a little confusing. When "converse" is used as a noun, we often say "... is the converse *of* ...'

But when "converse" is used as an adjective, we often say "... is converse *to* ..."

What we have witnessed is: **A statement can be true even though its converse is false. A statement can be false even though its converse is true.**

Example 12.1.3. If a shape is a square, then it has four sides. (True statement.)

The converse statement is: If a shape has four sides, then it is a square. (False statement. There are four-sided shapes like parallelograms and rectangles that may not be squares!)

12.2 Differentiability

Today and tomorrow, we're going to talk about something called the *Mean Value Theorem*. But we need to build up to it first. Like most mathematically meaningful statements, a theorem will be an "if-then" statement. (It will have a hypothesis and a conclusion.) But we need some new words to talk about the hypothesis, and to gain an appreciation for it.

Definition 12.2.1. Let f be a function, and x a number. We say that *f is differentiable at x* if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

This limit should look familiar. It's the thing we write down to define and compute the derivative! I've mentioned a couple times that *when* the secant lines through x and $x+h$ converge to a single line, then we'll call the resulting single line is called the tangent line to f at x. And we'll call its slope the derivative.

But we haven't really looked at examples where the "when" fails. That is, we haven't looked at examples of functions where a tangent line might not exist at x!

Let's see such an example. It's pretty much the only example you'll need to know, at least for this course.

Example 12.2.2. Let $f(x) = |x|$.

What does the graph of this function look like? Well, remember that the absolute value function just returns the "size" of a number, or the distance of that number from 0. For example, $|5|$ is just 5. And $|-5|$ is also 5. Another way to think about absolute value is that it converts every number to its "positive form." So if a number is already positive, the absolute value just returns that number, while if a number is negative, the absolute value makes it positive.

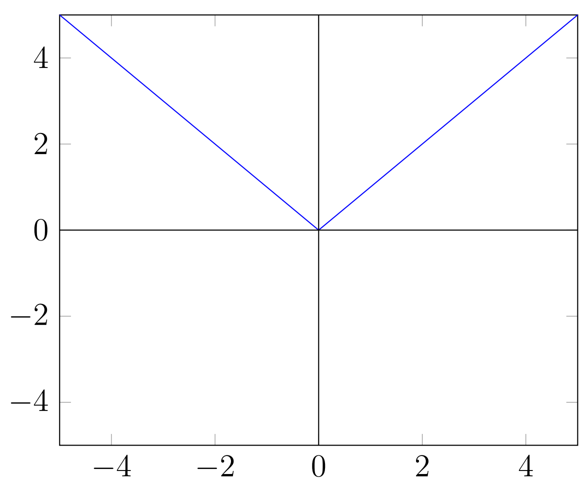
One way to express this is to use what's called a *piecewise* notation for the function:

$$f(x) = \begin{cases} x & x > 0 \\ -x & x < 0 \\ 0 & x = 0. \end{cases}$$

In other words, f sends a number x

- to itself if $x > 0$, or
- to negative of itself if $x < 0$, or
- to zero if $x = 0$.

Note that the graph of f will thus consist of two parts: The function $f(x) = x$ when x is positive, the function $f(x) = -x$ when x is negative, and the point telling us that $f(0) = 0$. All told, the graph (drawn in blue) looks like this:



I claim that $f(x) = |x|$ *does not have a tangent line at* $x = 0$.

Indeed, if you look at the secant line through x and $x+h$ for positive h, the secant line is a line of slope +1. But any secant line through x and $x+h$ for negative h is a line of slope -1.

So there's no way that a line of slope +1, and a line of slope -1, will converge to a single line as h approaches zero.

Put another way, if we have a function $g(h)$ that assigns +1 to h whenever $h > 0$, but assigns -1 whenever $h < 0$, then there's no value that $g(h)$ "wants to approach" as h approaches zero. So the difference quotient does not have a limit as $h \to 0$.

In sum, we have see that

The function $f(x) = |x|$ is not differentiable at $x = 0$.

In other words, the function does not have a derivative at $x = 0$.

12.2.1 The Mean Police

Let's begin with a story about the Mean Police. A college friend told me that this he watched a video about the Mean Police to learn about the Mean Value Theorem.

One day, you drive from Austin to San Antonio, which is about an 80-mile drive. You make the drive in 50 minutes.

Then one day, the Mean Police come knocking on your door. They say that they know you drove 80 miles in 50 minutes. So, by their math,

$$\frac{80 \text{ miles}}{50 \text{ minutes}} = \frac{80 \text{ miles}}{\frac{5}{6}\text{hours}} = \frac{80}{\frac{5}{6}} \text{ miles per hour} = \frac{480}{5} \text{ miles per hour} = 96 \text{ miles per hour}$$

was the speed at which you drove.

You retort: No, no. There's no way I was driving 96 miles per hour that whole time! I began in Austin, and I couldn't have been driving 96 miles per hour through that city.

The Mean Police respond: The speed limit is 80 miles per hour at the fastest; and we know that you were driving *at least* 96 miles per hour at some point. So that's enough for us to issue you a ticket.

Of course, you don't want a ticket, but are the Mean Police correct? It's not like they even had a radar gun to measure your speed. Were you driving at least 96 miles per hour at some point?

The math The Mean Police are correct, of course. They don't know when exactly you were driving above the speed limit, and they don't know where, either. But if you could make an 80-mile trip in 50 minutes, at *some point* you were driving at least 96 miles per hour. And there's no place in Texas where the speed limit is that high. So you were breaking the speed limit at some point!

And this basic math is quite powerful. From the perspective of the Mean Police, they didn't need to set up radar guns or have police stake out strips of highway to measure your speed; all they needed to know were your starting point, ending point, and the length of time you traveled.

Let's translate this into math language. Suppose f is the function that ways how far you've traveled, where f is in units of miles, and the input will be in units of hours. (So at time t hours into your trip, you've traveled $f(t)$ miles.)

What the police know are that

$$f(0) = 0 \qquad \text{(because you haven't traveled any miles when you begin, 0 hours into your trip)}$$

and

$$f(5/6) = 80 \qquad \text{because you traveled 80 miles in 5/6 of an hour.}$$

And what they conclude is that *at some time* t, you were traveling at 96 miles per hour. In other words, the police are claiming that at some time t, they know that

$$f'(t) = 96.$$

Remember, 96 came from measuring

$$\frac{f(5/6) - f(0)}{5/6} = 96.$$

To summarize: If we know where you began, where you ended, and how long it took you, we *don't* know how exactly you drove, but we *do* know that you reached a certain speed at some point.

12.2.2 Mathematical translation

So let's suppose that f is some function. In the example above, f could be a function that takes the time as an input, and outputs where you are.

What we know is

1. A starting time a,
2. An ending time b,
3. Where you began, $f(a)$, and

4. Where you ended, $f(b)$.

Based on this information, we can conclude: At some point between a and b (inclusive), you had a speed of $(f(b)-f(a))/(b-a)$. In other words, there is some moment c in the interval $[a, b]$ such that $f'(c) = (f(b) - f(a))/(b - a)$.

We'll state this as a theorem:

Theorem 12.2.3 (Mean Value Theorem)**.** Let f be a function, and choose two numbers a and b for which $a < b$. Suppose that f is differentiable at every point between a and b. Then there exists some number c in the interval $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 12.2.4. At noon yesterday, the temperature in San Marcos was 60 degrees Farenheit. At noon today, the temperature was 62 degrees Farenheit. Over those 24 hours, the temperature changed by a total of +2 degrees Farenheit. So, by the Mean Value Theorem (assuming that temperature is a differentiable function of time), we know there was *some moment* in those 24 hours at which the temperature was changing at a rate of $\frac{2}{24}$ degrees per hour. That is, a rate of 1/12 degrees per hour.

(That does not mean that the temperature changed 1/12 degrees in some hour; it means at some *moment* in time, 1/12 deg/hr is how fast the temperature was changing.)

In this example, c is that moment, a is noon yesterday, b is noon today, $f(a)$ is 60, $f(b)$ is 62, and $f(x)$ measures the temperature at time x (measured in hours).

Expectation 12.2.5. You are expected to read and to understand the statement of Theorem 12.2.3 above.

12.3 Studying the Mean Value Theorem

We can write the Mean Value Theorem out as an if-then statement. You should try to do this on your own, too, but here's what we would get. Remember that if-then statements have a hypothesis and a conclusion: "If HYPOTHESIS, then CONCLUSION."

In the Mean Value Theorem, the hypotheses are these:

- f is a function[1]

[1] Technically, f needs to be continuous, too, but we're going to ignore this necessary hypothesis for now.

- a and b are two numbers with $a < b$, and
- f is differentiable at every number between a and b.

Then the *conclusion* is:

- There is some number $a < c < b$ so that the derivative of f at c is equal to the value
$$\frac{f(b) - f(a)}{b - a}.$$

In the real world, you can often ignore the hypothesis that "f must have a derivative at every point between a and b," because this is often true or assumed. But in math, we often have functions like $|x|$ that don't have derivatives at certain points, so the hypothesis is quite important.

Example 12.3.1. Let $f(x) = x\sin(x)$. Note that $f(0) = 0$, and that $f(\frac{\pi}{4})$ is equal to
$$\frac{\pi}{4}\sin(\frac{\pi}{4}) = \frac{\pi}{4} \cdot 1 = \frac{\pi}{4}.$$
Setting $a = 0$ and $b = \frac{\pi}{4}$, we can compute that
$$\frac{f(b) - f(a)}{b - a} = \frac{\frac{\pi}{4} - 0}{\frac{\pi}{4} - 0} = 1.$$
So, at some point between $a = 0$ and $b = \frac{\pi}{4}$, there is a number c for which
$$f'(c) = 1.$$
And we know what f' is—the derivative of f is the function $f'(x) = \sin(x) - x\cos(x)$. So, in other words, there is some number c between 0 and $\frac{\pi}{4}$ so that
$$\sin(c) - c\cos(c) = 1.$$
At first glance, the equation above is kind of subtle! Is it obvious that there should be a number so that $\sin(c) - c\cos(x)$ is equal to 1?

Well, the Mean Value Theorem doesn't tell us where c is, but it does tell us that c exists.

Example 12.3.2. Let $f(x) = x^3$. Let's note that $f(1) = 1$ and $f(1.1) = 1.331$. Then
$$\frac{f(1.1) - f(1)}{1.1 - 1} = \frac{1.331 - 1}{0.1} = \frac{0.331}{0.1} = 3.31.$$

So, using the Mean Value Theorem, we know that there is some number c between 1 and 1.1 that satisfies the equation $f'(c) = 3.31$.

Well, what is f'? We know that $f'(x) = 3x^2$. So if we were to try to solve the equation $f'(x) = 3.31$, we would see the following:

$$\begin{aligned} f'(x) &= 3.31 \\ 3x^2 &= 3.31 \\ x^2 &= \frac{3.31}{3} \\ x &= \pm\sqrt{\frac{3.31}{3}} \end{aligned} \tag{12.3.1}$$

It's probably not obvious at first glance whether the square root of $(3.31)/3$ is between 1 and 1.1. But the Mean Value Theorem tells us that it must be (because, by solving the equation above, we see that this is the *only* number at which the derivative of f is equal to 3.31).

So this isn't a typical use of the Mean Value Theorem, but it's something.

12.4 Some general applications of the Mean Value Theorem

12.4.1 Functions that aren't constant

Definition 12.4.1 (Constant functions)**.** We'll call a function *constant* if for every pair of input values, the outputs are the same. That is, f is constant if $f(a) = f(b)$ for every choice of a and b.

A less complicated way to visualize this is that the graph of f is a horizontal line.

But visualizations can be misleading, so I'm going to use the complicated definition.

Proposition 12.4.2. Suppose f is differentiable. If we know that f is *not* a constant function, then there is some c for which $f'(c)$ is not zero.

Proof. Suppose we know that f is not constant. That means that there are two different inputs that have different outputs. Let's call these inputs a and b, so that the outputs $f(a)$ and $f(b)$ are not equal.

Because f is differentiable, we can use the Mean Value Theorem. The theorem tells us that there is some input c for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

On the right, the numerator is not zero, because we know that $f(b) \neq f(a)$. This means that the fraction on the righthand side is some fraction with a non-zero number in the numerator. Such fractions never equal zero. So $f'(c)$ is not zero.

That finishes the proof! □

12.4.2 There are many ways to write a proof for a statement

As with any good medium of expression, there are *many* ways to write a proof. But the *content* must be solid. Below are some examples.

By the way, the white box at the end means "end of proof." Sometimes, people also write "QED" (quod erat demonstrandum) at the end of a proof.

Proof. Because f is not constant, we can find two numbers $a \neq b$ with $f(a) \neq f(b)$. We can note

$$\frac{f(b) - f(a)}{b - a}$$

is a number that does not equal zero, because the numerator is not zero.

On the other hand, the Mean Value Theorem says that (because f is differentiable) there is some number c satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The righthand side—as we just said—is not zero. Thus $f'(c) \neq 0$. Putting everything together, we see that there is some point c at which $f'(c) \neq 0$. □

Proof. By the Mean Value Theorem, for any two points a and b, we know there is some c in $[a, b]$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

So we are finished if we can find a, b so that $f(b) - f(a)$ does not equal zero (for then $f'(c)$ will be non-zero). Well, we know that a and b satisfying $f(b) - f(a)$ exist because $f$is non-constant. QED □

Proof. We know the following:

1. There are two numbers a and b so that $f(b) - f(a) \neq 0$.

2. For some number c between a and b,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3. The fraction $\frac{f(b)-f(a)}{b-a}$ does not equal zero.

The first statement is true because f is not a constant function. The second statement is true by the Mean Value Theorem (which we can use because f is differentiable). The third statement is true because the fraction has a non-zero numerator (by the first statement).

Putting 2. and 3. together, we have found a number c so that $f'(c)$ does not equal zero. QED. $\square$