# Lecture 10

# Second derivatives, concavity, and minima/maxima

## 10.1 Second derivatives

Today, we will practice taking "second derivatives," and knowing when they are positive or negative.

**Definition 10.1.1.** The second derivative of $f$ is *the derivative of the derivative*[1] of $f$. We denote the second derivative by

$$f'', \qquad \text{or} \qquad \frac{d}{dx}(\frac{d}{dx}f), \qquad \text{or} \qquad \frac{d^2}{dx^2}f, \qquad \text{or} \qquad \frac{d^2f}{dx^2}. \tag{10.1.1}$$

**Example 10.1.2.** Let $f(x) = 3x^2 + x - 7$. Then the (first) derivative of $f$ is

$$f'(x) = 6x + 1.$$

If we take the derivative of $f'(x)$, we end up with the second derivative of $f$:

$$f''(x) = 6.$$

**Example 10.1.3.** Here are more examples of functions and their second derivatives. You should verify these examples:

- If $f(x) = \sin(x)$, then $f''(x) = -\sin(x)$.

[1] Yes, there are two appearances of the word "derivative"; this is not a typo.

- If $f(x) = e^x$, then $f''(x) = e^x$.
- If $f(x) = e^{5x}$, then $f''(x) = 25e^{5x}$.
- If $f(x) = x^3 - 5x^2$, then $f''(x) = 6x - 10$.

**Example 10.1.4.** Let's find the second derivative of $f(x) = \ln(x)$. As defined above, we just need need to take the derivative twice. Let's take the first derivative:

$$f'(x) = \frac{1}{x}.$$

(This is something we learned in class.) Now let's take another derivative—for example, by using the quotient rule—to find

$$f''(x) = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}.$$

That is, the second derivative of $\ln x$ is $-1/(x^2)$.

**If you know how to take derivatives, you know how to take second derivatives.** So you see how our skills are building on each other—make sure you practice taking derivatives!

**Example 10.1.5.** Let $f(x) = x^2 - 2$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 2x$$

so

$$f''(x) = 2.$$

So the second derivative is always 2, meaning the second derivative is positive *everywhere.*

**Example 10.1.6.** Let $f(x) = x^3 - 3x^2 + 3$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 3x^2 - 6x$$

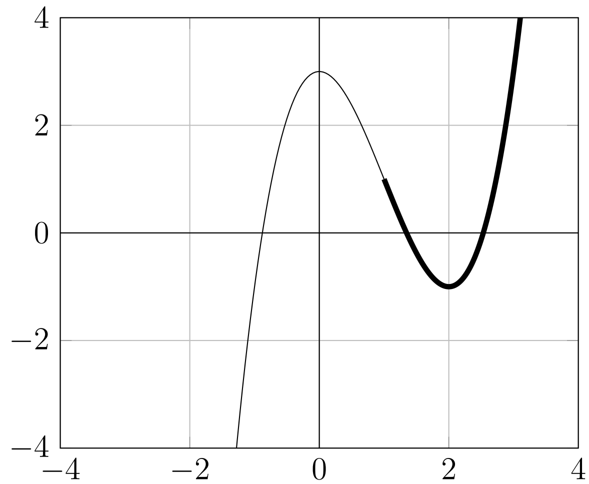
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = 6x - 6.$$

So the second derivative is positive when $6x - 6$ is positive. This happens exactly when $6x > 6$—that is, when $x > 1$.

As a bonus: The second derivative is negative when $6x < 6$—that is, when $x < 1$.

Below is a graph of $f(x)$, and I have shaded in **bold** the part of the graph where the second derivative is positive:




**Example 10.1.7.** Let $f(x) = x^4 - 24x^2 + 50$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 4x^3 - 48x$$

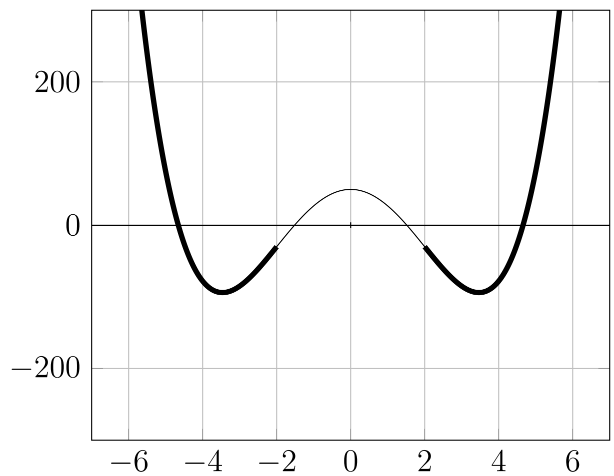
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = 12x^2 - 48.$$

So the second derivative is positive when $12x^2 - 48$ is positive. This happens exactly when $12x^2 > 48$—that is, when $x^2 > 4$. But $x^2 > 4$ exactly when $x < -2$ or $x > 2$.

As a bonus: The second derivative is negative when $x^2 < 4$—that is, when $x$ is between $-2$ and $2$.

Below is a graph of $f(x)$, and I have shaded in **bold** the part of the graph where the second derivative is positive:



**Example 10.1.8.** Let $f(x) = 3\sin(x)$. *Where is the second derivative positive?*

Let's find the second derivative. We see that

$$f'(x) = 3\cos(x)$$

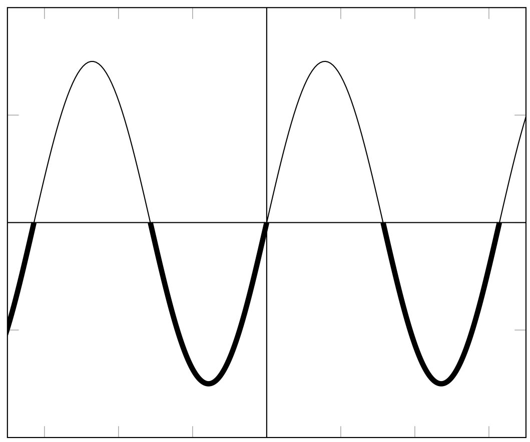
so, taking the derivative of $f'(x)$, we find:

$$f''(x) = -3\sin(x)$$

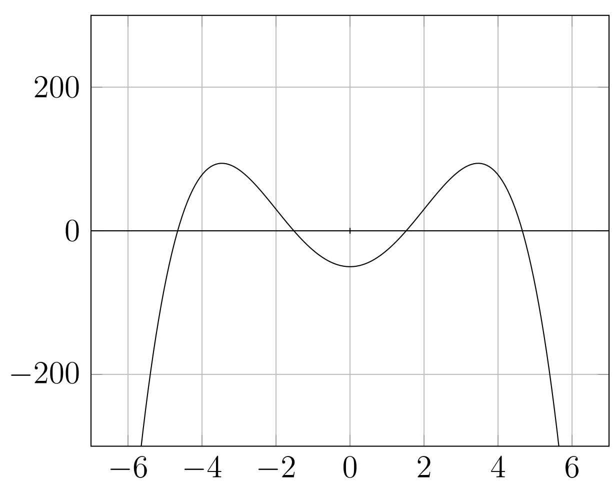
So the second derivative is positive when $-3\sin(x)$ is positive. This happens exactly when $\sin(x)$ is negative. And based on our trigonometry knowledge from precalculus, we know that this happens when

- $x$ is between $\pi$ and $2\pi$,
- $x$ is between $3\pi$ and $4\pi$,
- $x$ is between $-\pi$ and $0$,
- $x$ is between $-3\pi$ and $-\pi$,
- $\dots$.

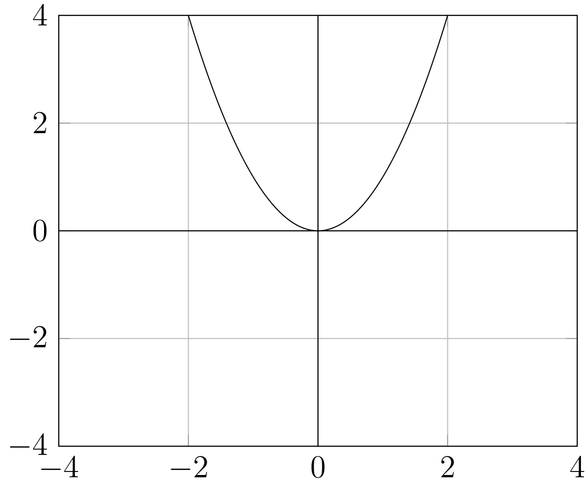
Below is a graph of $f(x)$. I have shaded in **bold** the part of the graph where the second derivative is positive:



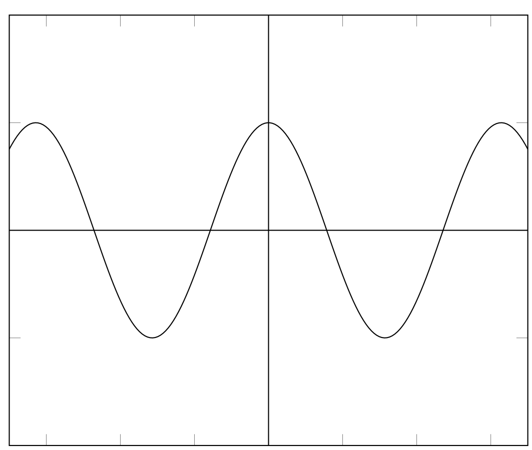
For next class, I expect you to be able to do the following: For each of the functions $f(x)$ below, (i) State *where* the function has a *positive* second derivative, and (ii) Shade in **bold** where the graph of the function has positive second derivative. (You will be provided the graph of $f(x)$.)



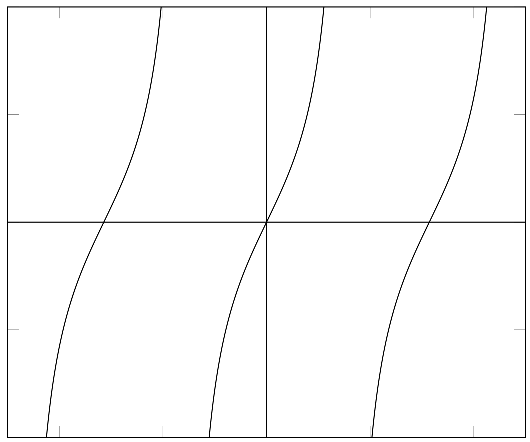

(a) $f(x) = -x^4 + 24x^2 - 50$.




(b) $f(x) = x^2$.



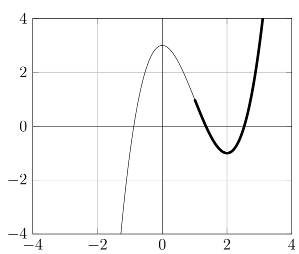
(c) $f(x) = \cos(x)$.



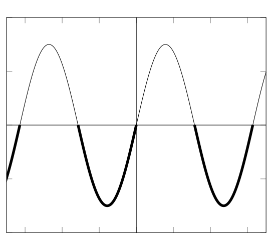
(d) $f(x) = \tan(x)$.

You have seen examples of graphs with positive second derivative. Here are some examples, with the positive-second-derivative regions shaded in *bold*:

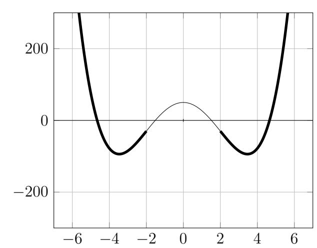
1. $f(x) = x^3 - 3x^2 + 3$:



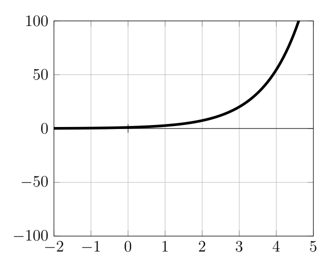

2. $f(x) = 3\sin(x)$:



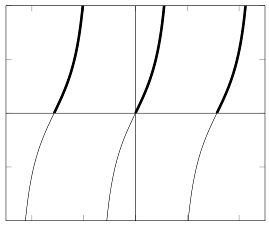
3. $f(x) = x^4 - 24x^2 + 50$:




4. $f(x) = e^x$:




5. $f(x) = \tan(x)$:



## 10.2 Concavity

The point I want to make with these pictures is that *the value of the second derivative gives us some idea of what the graph looks like.* (Though not a complete picture.)

Intuition: On the regions where the second derivative is positive, the graph of $f$ looks like a *portion* of an "upright bowl." Some students have described this as "opening upward" as well.

Conversely, when the second derivative is negative, the graph of $f$ looks like a portion of an "upside-down bowl." But we have technical names, too. From now on, you are expected to know the following terminology:

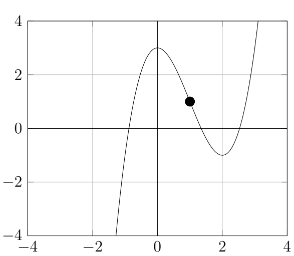
**Definition 10.2.1** (Concavity)**.** We say that $f$ is *concave up* at $x$ if $f''(x) > 0$. We say that $f$ is *concave down* at $x$ if $f''(x) < 0$.

## 10.3 Inflection points

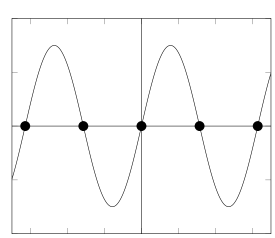
**Definition 10.3.1.** If $f''(x) = 0$, and the concavity of $f$ *changes* at $x$, we say that $x$ is an *inflection point.*

**Example 10.3.2.** Here are some examples of functions and their graphs, with their inflection points labeled.

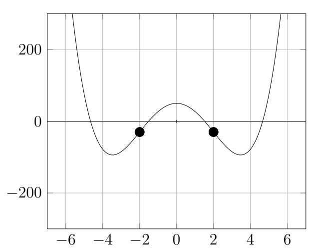
1. $f(x) = x^3 - 3x^2 + 3$:



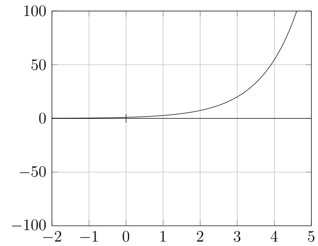

2. $f(x) = 3\sin(x)$:



3. $f(x) = x^4 - 24x^2 + 50$:

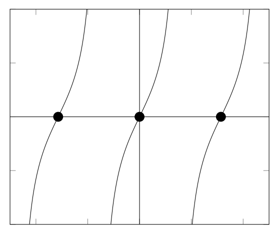



4. $f(x) = e^x$:

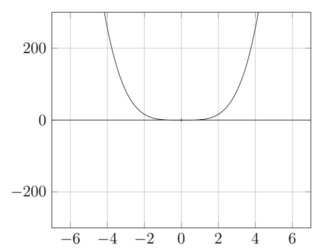



(No inflection points.)

5. $f(x) = \tan(x)$:



6. $f(x) = x^4$:




(No inflection points, even though $f''(x) = 0$ at $x = 0$.)

**Expectation 10.3.3.** Based on looking at a graph, you are expected to be able to identify inflection points—an inflection point is a place at which a function switches concavity (from up to down, or from down to up).

**Exercise 10.3.4.** Now we're going to try to learn something about a function by knowing its derivative and second derivative. You can hunt for examples on the previous pages of this packet. Or, you can try understanding $f(x) = x^2$ and $f(x) = -x^2$.

1. Can you find an example of a function $f$, and a point $x$, where $f'(x) = 0$ and $f$ is concave up at $x$? What does the function $f$ look like near $x$? How does the value of $f$ at $x$ compare to the value of $f$ at nearby points?

2. Can you find an example of a function $f$, and a point $x$, where $f'(x) = 0$ and $f$ is concave *down* at $x$? What does the function $f$ look like near $x$? How does the value of $f$ at $x$ compare to the value of $f$ at nearby points?

## 10.4 For next time

You should be able to tell me the second derivatives of the following functions:

(a) $x^3 - 3x^2 + x$

(b) $4x^2 + 3x - 2$

(c) $e^{7x}$

(d) $\sin(x)$

You should also be able to tell me where the following functions are concave up:

(a) $x^3 - 3x^2 + x$

(b) $4x^2 + 3x - 2$

(c) $e^{7x}$