Lecture 8

Derivatives of other inverse functions. And, why e?

8.1 Inverses, revisited

Last time, we learned about two derivatives: The first was the derivative of $f(x) = e^x$. We learned that

$$\text{If } f(x) = e^x, \text{ then } f'(x) = e^x.$$

That is, e^x is some seemingly special function—it is its own derivative!

Based on this fact (which we took for granted), we learned about *inverses*, and learned that we can try to compute derivatives of inverse functions. As an example, we recalled that $\ln x$ is an inverse to e^x, and we deduced that

$$\text{If } f(x) = \ln x, \text{ then } f'(x) = \frac{1}{x}.$$

But let's talk a little bit about what an inverse function is. I am going to ignore the words "right" and "left" for today, to simplify things.

Informally, an *inverse to f* is a function that "undoes" f. For example, f takes a number x, and outputs a number called $f(x)$. What does it mean to undo this? Well, to undo this process would be to take a number called $f(x)$, and output/return a number called x.

Example 8.1.1. If $f(x) = e^x$, f takes a number, then outputs e to that number. For example, f takes a number like 2, and outputs a number e^2, which is roughly $7.38905609893\ldots$.

If there is to be a function g that applies *undo* to f, it must take the number $7.38905609893\ldots$ and output 2. More accurately, if g sees an input called e^2, it ought to return 2. And more generally, if g sees an output balled e^{blah}, g should output blah.

The great thing is that you had already seen such a function in precalculus—this function is called ln, or the natural log.

You have seen other examples of inverses. For example, sin is a function that takes in an *angle*, and outputs a *height* (of a point on the unit circle). Do you think we could go backward? For example, if we are given a *height* of a point on the unit circle, we might be able to say what angle that point is at.

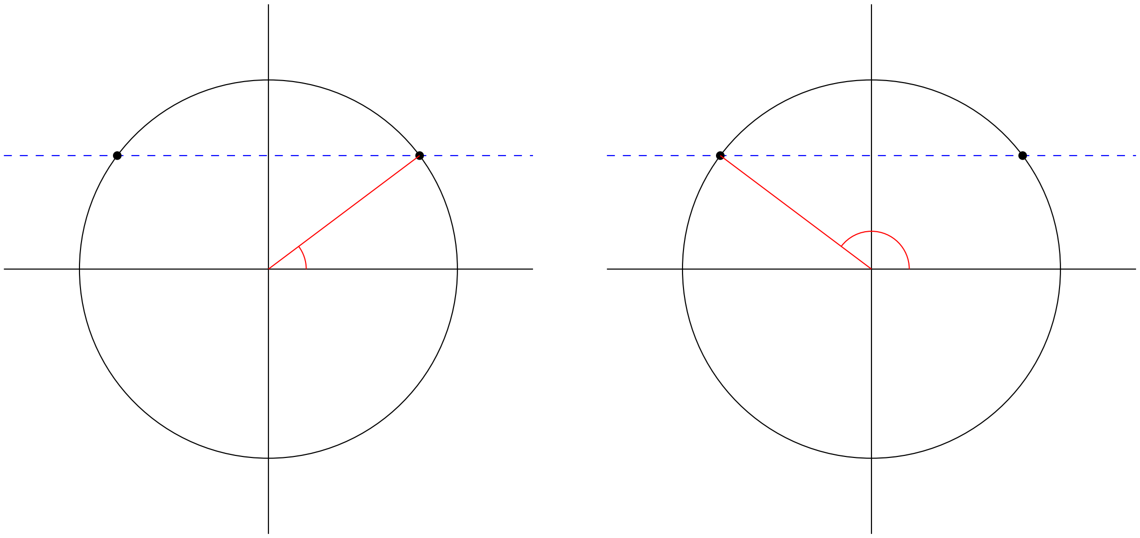

Figure 8.1: A single height (the blue dashed line) determines two possible points (the black dots) on the circle, hence two possible angles (in red).

Above is a picture of a blue dashed line (drawn to indicate, for example, a line of height 0.6). We see an immediate issue, which is that the blue dashed line (i.e., a height) actually defines *two* possible points on the circle. So it's not clear which angle we should take. See Figure 8.2.

So let's just *agree* as a community that, if we want to specify a point or an angle from a height, we will always take the point or angle on the *right half* of the unit circle. We will call this angle the *arcsine*, or *inverse sine*, of the height.

$$1 = (\sin'(\arcsin(x)) \cdot \arcsin'(x)). \qquad = (\cos(\arcsin(x)) \cdot \arcsin'(x)). \tag{8.1.1}$$

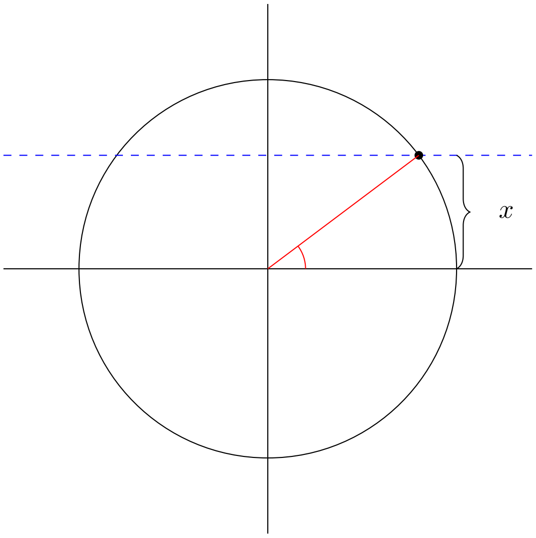

Figure 8.2: The red angle is $\arcsin(x)$ (in radians).

so

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

8.2 What's up with e?

I don't know how you were introduced to the number e, but let's talk about a really cool reason to care about e.

First, let's consider the following functions:

1. $f(x) = 2^x$
2. $f(x) = e^x$
3. $f(x) = 3^x$
4. $f(x) = 5^x$.

You know how to take the derivatives of these functions. For example, to take the derivative of 2^x, you might write

$$2^x = e^{(\ln 2)\cdot x}$$

so

$$(2^x) = \ln 2 e^{(\ln 2)\cdot x} = \ln 2 \cdot 2^x.$$

In other words, when $f(x) = 2^x$, we see that

$$f'(x) = \ln 2 \cdot 2^x = \ln 2 \cdot f(x).$$

Taking the derivatives of the other functions, we see

1. $f(x) = 2^x \implies f'(x) = \ln 2 f(x)$
2. $f(x) = e^x \implies f'(x) = f(x)$
3. $f(x) = 3^x \implies f'(x) = 3f(x)$
4. $f(x) = 5^x \implies f'(x) = 5f(x)$.

So e is quite a special number! In fact, it's the *only* number such a that the derivative of a^x is equal to a^x itself.

That's what's so "natural" about e, and why we call ln, or log base e, the "natural log."[1]

8.3 The derivative of e^x

Last time I just claimed that the derivative of e^x is itself. How might we see that?

First, let $f(x) = a^x$, where a is some number. (It could be 2 or 3, but let's ignore what number it is exactly so that we can see a pattern.)

Then as usual, the derivative of f is computed by taking the limit of the difference quotient:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

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[1] By the way, you might have wondered why "natural log" is written ln as opposed to *nl*. Well, ln comes from the French, *logarithme naturel*, which you might guess means natural logarithm. But just like in Spanish, the order of the adjective and noun are flipped. (In Spanish, it's *logaritmo natural*.) Hence the ln, as opposed to nl.

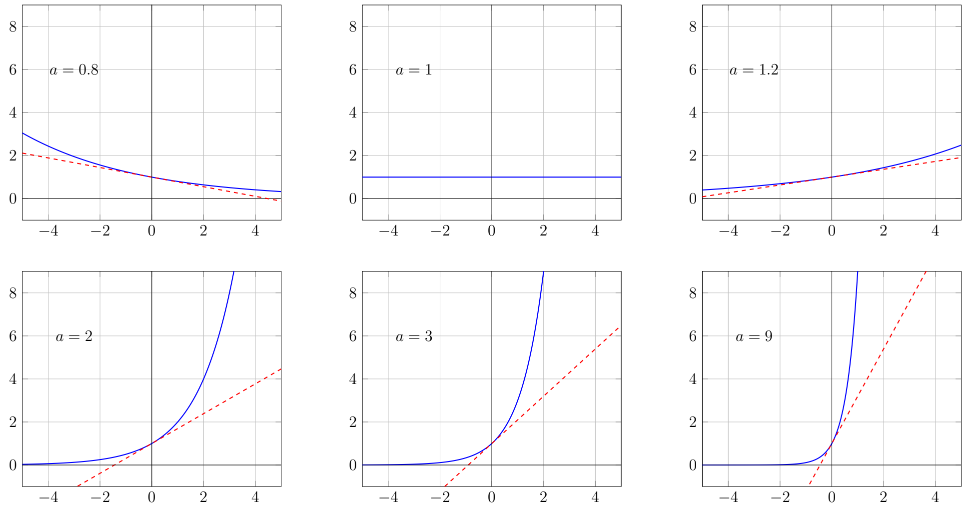
Plugging what f is, we find

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h\to 0} a^x \frac{a^h - 1}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h a^0 - a^0}{h} \\
&= a^x \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} \\
&= a^x \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\
&= a^x f'(0).
\end{aligned} \tag{8.3.1}$$

In other words, the derivative of a^x is always given by *the value of* a^x *times the derivative of* a^x *at zero.*

In other words, the derivative of a^x is pretty much the same thing as a^x, but scaled by whatever the derivative at $x = 0$ is.

We can draw the graphs of $f(x) = a^x$ for different values of a:



The tangent line at $x = 0$, for each value of a, is drawn. Note that the tangent line has negative slope at $a = 0.8$ (when $a < 1$), is flat—and hence has slope zero—when $a = 1$, then the slope keeps getting positive, and bigger and bigger, as a increases.

Thus, for *some* value of a, the slope must equal exactly 1!

And why does that matter? Well, for that value of a, we thus have that $f'(0) = 1$. Hence for that value of a, $f'(x) = f(x)$.

You can *define* e to be the value of a for which the limit of $(a^h - 1)/h$ as $h \to 0$ is given by 1. This is probably the craziest way you've ever seen a number defined, and it really takes a very clever person to think up of the *existence* of such a number without constructing it. But indeed, we have done this as a civilization, and we can now utilize it.

8.4 For next time

You should be comfortable with finding the derivatives of the following functions:

(a) $f(x) = \arcsin(x)$

(b) $f(x) = e^{\arcsin(x)}$

(c) $f(x) = \arcsin(1 + x)$

(d) $f(x) = \ln(x^2 + 1)$

(e) $f(x) = \ln(\arcsin(x))$.