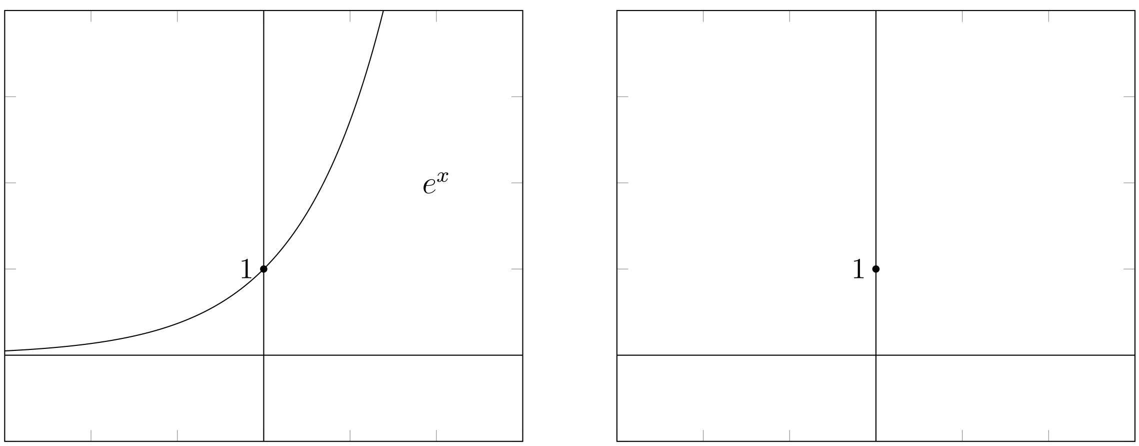
Lecture 7

Derivatives of exp and ln

7.1 Drawing derivatives of a graph

Exercise 7.1.1. Below on the left is the graph of $f(x) = e^x$.



Let me tell you the following fact: The derivative of e^x at $x = 0$ is 1. (In fact, the value of e^x at $x = 0$ is 1 also.)

(a) Based on this, draw the derivative of e^x on the right.

(b) How does your drawing compare to the graph of e^x?

In fact, we have the following theorem:

Theorem 7.1.2 (Derivative of e^x)**.** The derivative of e^x is itself. That is,

$$(e^x)' = e^x.$$

Put another way,

$$\frac{d}{dx}(e^x) = e^x.$$

How cool is that? There's a function that is its *own* derivative!

Example 7.1.3. Let's find the derivative of e^{3x}. We have

$$\frac{d}{dx}(e^{3x}) = \frac{d}{dx}(3x) \cdot \frac{d(e^x)}{dx}(3x) \tag{7.1.1}$$

$$= 3 \cdot e^{3x}. \tag{7.1.2}$$

We have used the chain rule in the first line. If you're confused by it, it may be worthwhile to write this out step-by-step. Let's let $f(x) = e^x$ and $g(x) = 3x$. Then $e^{3x} = f(g(x))$. Thus

$$\frac{d}{dx}(e^{3x}) = \frac{d}{dx}f(g(x)) \tag{7.1.3}$$

$$= f'(g(x)) \cdot g'(x) \tag{7.1.4}$$

$$= f'(3x) \cdot g'(x). \tag{7.1.5}$$

(The second equality is due to the chain rule.) But we know that $f'(x) = e^x$ by Theorem 7.1.2, and we know $g'(x) = 3$ from previous lectures. Hence we can continue:

$$f'(3x) \cdot g'(x) = e^{3x} \cdot 3 = 3e^{3x}.$$

Exercise 7.1.4. Fix a real number B. Prove that the derivative of

$$f(x) = e^{Bx}$$

equals

$$Bf(x).$$

More generally, if you have another real number A, let

$$g(x) = Ae^{Bx}.$$

(For example, if you choose $A = 3$ and $B = 5$, you would have $3e^{5x}$. The previous example is when $A = 1$.) Prove that

$$g'(x) = Bg(x).$$

Application 7.1.5. This kind of behavior is incredibly important for *modeling*. For example, how fast is a population growing? In ideal circumstances, the more individuals there are in a population, the faster we expect the population to grow. Better yet, we might expect that the rate of population growth is *proportional* to the population itself! (Note that "being proportional to" is a far more precise relationship than "the bigger the population, the faster the growth".)

That's exactly what Exercise 7.1.6 tells us about $g(x) = Ae^{Bx}$. We see that g' is proportional to g (with proportional constant B). So for example, x could model time, while $g(x)$ could model the population at time x.

By the way, why might $g(x)$ be a *bad* model for population growth? For what kinds of situations might $g(x)$ be a *good model*? In those situations, what might A and B represent?

Exercise 7.1.6. Find the derivative of $f(x) = 5^x$. Hints: Remember that $5 = e^{\ln 5}$, remember the basic rules for dealing with exponents, and use the chain rule.

Exercise 7.1.7. Your friend is excited about the idea that $f(x)$ could equal $f'(x)$ and looks for more examples that looks like e^x. They try $f(x) = 5^x$, and are disappointed that $f'(x)$ does not equal $f(x)$.

Is it possible to find any number k—other than e—so that if $f(x) = k^x$, then $f'(x) = f(x)$?

Remark 7.1.8. Isn't e special?

Exercise 7.1.9. Now that you know the derivative of $g(x) = e^x$, can you figure out the derivative of $f(x) = \ln x$?

Hint: What is $g \circ f$? What if you try computing $(g \circ f)'$ using the chain rule, too?

7.2 Review of right inverses

Let f be a function. Here's a question: Given a value of f, can we always determine which x it came from?

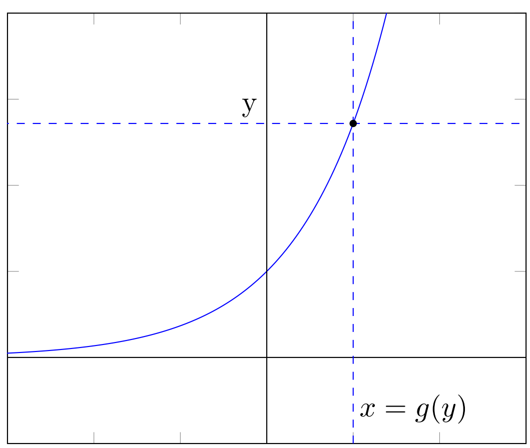
Example 7.2.1. Here are some examples:

1. If $f(x) = 3x$, and if someone tells you that f takes the value 12, you know exactly where: x must equal 4. In fact, in general, if f takes value y, you know the original x is $y/3$.

2. If $f(x) = 2^x$, and if someone tells you that f takes the value 8, you know exactly where: x must equal 3. In fact, in general, if f takes value y, you know f does so at $\log_2 y$.

3. If $f(x) = x^2$, and if someone tells you that f takes the value 4, you *don't* know exactly where: x could equal 2 or -2. However, *if* you restrict yourself to looking only for positive values of x, then if f takes value y, you know that the original x is $\sqrt{y}$.

Below is a visual way to think about this process. Drawn is the graph of f. Given a value y, can you figure out which value of x satisfies the equation $f(x) = y$? If so, that means that the coordinate x now becomes a function of y—you input y, and you output x—and we can call this function g.



Warning 7.2.2. While we were diligent about drawing g as a function of y before, from now on, we must now be comfortable realizing that letters are just letters, and we don't care if g takes inputs to be symbols that look like "x," or symbols that look like "y"; that is, g will often be treated as a function of x, too.

Definition 7.2.3. Let f be a function. We say that a function g is a *left inverse* to f if

$$(g \circ f)(x) = x.$$

Put another way, g "remembers" the original value x that outputted $f(x)$.

We also say that f is a *right inverse* to g. Put another way, f "knows" that if $g(\text{something}) = x$, then something $= f(x)$.

7.3 Derivatives of right inverses

It turns out that if we know the derivatives of a function g, then—if g has a right inverse f—we can figure out the derivatives of the right inverse f.

Lemma 7.3.1. Let g be a function, and suppose that f is a right inverse to g, defined on some open interval containing x. Suppose also that g is differentiable at $f(x)$, and that $g'(f(x)) \neq 0$. Then

$$f'(x) = \frac{1}{g'(f(x))}. \tag{7.3.1}$$

That is, the derivative of f at x is computed by dividing 1 by the derivative of g *at* $f(x)$.

Proof. Let's look at the following string of equalities:

$$1 = (x)' \tag{7.3.2}$$
$$= (g \circ f)'. \tag{7.3.3}$$

The first equality is our knowledge of the derivative of the function x. The next equality is using the hypothesis that g is a right inverse to f, so that $f \circ g = x$.

In total, what this string of equalities says is that the function on the righthand side is equal to the (constant!) function on the lefthand side. So let's evaluate at some point x. We have

$$1 = (g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

We can divide both sides by $g'(f(x))$ so long as this number isn't zero; so we find:

$$\frac{1}{g'(f(x))} = f'(x) \qquad \text{when } g'(f(x)) \neq 0.$$

This is what we wanted. ∎

Example 7.3.2. $f = \ln(x)$ is a right inverse to $g(x) = e^x$. This is because

$$g \circ f(x) = e^{\ln x} = x.$$

We know the derivative of $g(x) = e^x$, so we can use the lemma to find the derivative of $f(x) = \ln(x)$! Let's try:

$$\begin{aligned}
(\ln(x))' &= f'(x) && (7.3.4)\\
&= \frac{1}{g'(f(x))} && (7.3.5)\\
&= \frac{1}{e^{f(x)}} && (7.3.6)\\
&= \frac{1}{e^{\ln(x)}} && (7.3.7)\\
&= \frac{1}{x}. && (7.3.8)
\end{aligned}$$

. The first equality is by definition of f. The next equality is using Lemma 7.3.1. The rest is just plugging in our knowledge of g' and ln.

7.4 The derivative of natural log

The example from the last page is important, so let's record this as a theorem. (You will be expected to know this:)

Theorem 7.4.1 (The derivative of ln)**.** The derivative of ln is "one over x." That is,

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

7.5 For next lecture

You should be comfortable finding derivatives of functions involves e^x and ln. For example, you should be able to find f' for each of the following functions f:

(a) $f(x) = e^x$

(b) $f(x) = e^{3x}$

(c) $f(x) = e^{3x+2}$

(d) $f(x) = 3e^x$

(e) $f(x) = 5^x$

(f) $f(x) = 5^{3x}$

(g) $f(x) = \ln(x)$

(h) $f(x) = \ln(3x)$

(i) $f(x) = \ln(x + 3)$