Review for final exam

① Derivatives

★ Def: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ ~ slope of the tangent line

★ Basic laws :

re

$(\text{constant})' = 0$

$(x)' = 1.$

$(kf)' = k.f'$ (k is a constant - Scale law)

$(f \pm g)' = f' \pm g'.$ (Addition / Subtraction)

$(f.g)' = f'g + fg'.$ (Product Rule)

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (Quotient Rule)

$(x^n)' = nx^{n-1}$ (Power Law)

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x.$

Chain Rule : $(f\circ g)'(x) = f'(g(x)).\, g'(x)$

Ex: $((\sin(x))^3)'$

$g(x) = \sin(x)$, $f\circ g(x) = f(g(x)) = \sin(x)^3$

$\Rightarrow (\sin(x)^3)' = 3\sin(x)^2.\cos x.$

$(e^x)' = e^x$, $(a^x)' = \ln a . a^x.$

$(\ln x)' = \frac{1}{x}$, $(\log_a x)' = \frac{1}{\ln a . x}.$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x.$

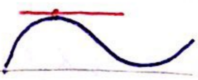
★ **Problem type: Find the derivatives of the functions.**

→ Make sure you remember all the basic laws & can use them to efficiently find the derivatives
(Tips: Practice with problems in the classnotes, lab worksheets).

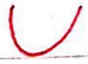
★ **Maximum & Minimum.**

Critical points ↗

$f'(x) = 0$ → Horizontal tangent line.



$f''(x) > 0$ → Concave up.

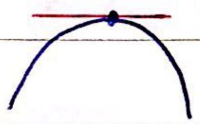


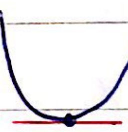
$f''(x) < 0$ → Concave down

$f''(x) = 0$ → Inflection point

Problem type: Find maximum, minimum? (local)

→ **Method 1:** Use second derivative test.





$\begin{cases} f'(x) = 0 \\ f''(x) < 0 \end{cases}$ → Maximum

$\begin{cases} f'(x) = 0 \\ f''(x) > 0 \end{cases}$ → Minimum

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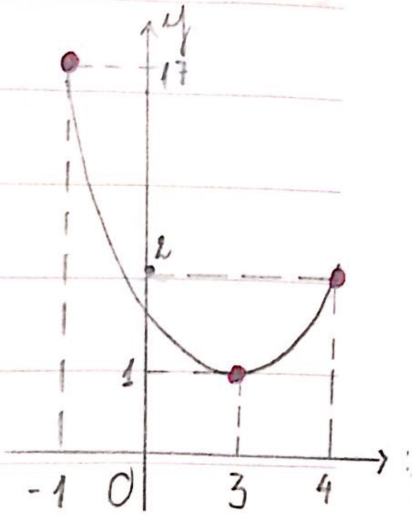
→ **Method 2:** If you find absolute max, min on an interval $[a, b]$.

- Step 1: Find $f'(x)$
  Find all critical points x_i by solving $f'(x) = 0$
- Step 2: Find the value of f at all critical points & 2 bounds of the interval (a, b).
  $f(x_1) \ldots f(x_n)$, $f(a)$, $f(b)$.
- Step 3: Compare the value to find max, min.

Ex 5 : $f(x) = x^2 - 6x + 10$ on $[-1, 4]$. Find max, min absolute.



$f'(x) = 2x - 6$

$f'(x) = 0 \Leftrightarrow 2x - 6 = 0 \Leftrightarrow x = 3.$

$f(3) = 1, \quad f(-1) = 17, \quad f(4) = 2.$

$\Rightarrow$ Absolute max : 17, at $x = -1$.

Absolute min : 1, at $x = 3$.

★ Implicit Differentation.

Ex 12 b) $4x^2 + 8 = xy^2$.

$8x + 0 = x'y^2 + x(y^2)'$ (Take derivatives of both sides, Remember y is a function of x)

$8x = y^2 + x \cdot 2y \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{8x - y^2}{2xy}$

c) Horizontal tangent line $\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow \frac{8x - y^2}{2xy} = 0 \Rightarrow 8x - y^2 = 0.$ (*)

We have $4x^2 + 8 = xy^2 \Rightarrow y^2 = \frac{4x^2 + 8}{x}$ (**)

Substitute (**) into (*):

$\Rightarrow 8x - \frac{4x^2 + 8}{x} = 0 \Rightarrow 8x^2 - 4x^2 - 8 = 0$

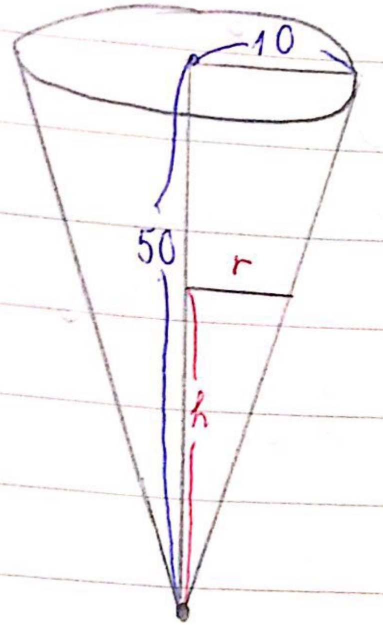
$\Rightarrow 4x^2 - 8 = 0 \Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$.

$\Rightarrow x = \sqrt{2}$ $(x > 0)$.

★ Related Rates.

Ex 11:



- Intepret information

+) Water flows out at a rate of 10 cubic feet per minute

→ $V'(t) = -10\ ft^3/min$

+) Q: How fast is the depth of water changing when the depth is 20 feet?

→ Find $h'(t)$ at $t = 20$.

$h'(20) = ?$

- Hint $V = \frac{1}{3}\pi r^2 h$.

What is the relation of r & h?

$\frac{r}{10} = \frac{h}{50} \Rightarrow h = 5r$ or $r = \frac{h}{5}$ $\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 \cdot h = \frac{\pi h^3}{75}$.

$\Rightarrow V'(t) = \frac{\pi}{75} \cdot 3h^2 \cdot h'(t) = \frac{\pi h^2 h'(t)}{25}$ (h is a function of t)

$\Rightarrow -10 = \frac{\pi h^2 h'(t)}{25}$

@ $t = 20$: $-10 = \frac{\pi \cdot 20^2 h'(20)}{25}$

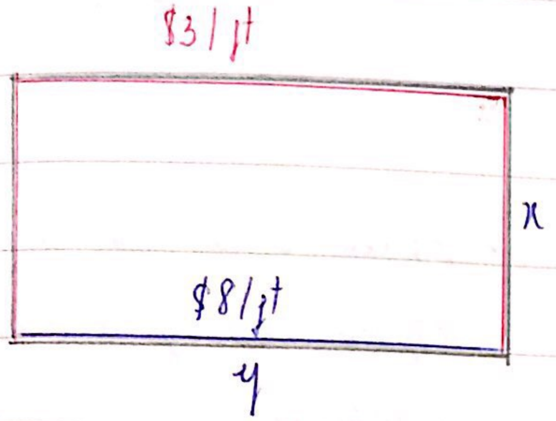
$h'(20) = -\frac{5}{8\pi} \approx -0.2$ (ft/min)

Remember the unit.

★ Real life optimization problems

Ex 10:

Step 1: Understand the context, build mathematical model.



Area = 2000 ⇒ $xy = 2000$ ⇒ $y = \frac{2000}{x}$

Total cost:

$f(x,y) = 3(2x+y) + 8y = 6x + 11y$

Substitute $y = \frac{2000}{x}$

⇒ Total cost:

$f(x) = 6x + \frac{11 \times 2000}{x} = 6x + \frac{22000}{x}$

Step 2: Solve the mathematical problem arised.
(Find the mininum).

$f'(x) = 6 - \frac{22000}{x^2}$.

$f'(x) = 0 \Rightarrow 6 - \frac{22000}{x^2} = 0 \Rightarrow x^2 = \frac{22000}{6}$

$\Rightarrow x = \sqrt{22000/6} \; (x > 0), \; y = \frac{2000}{x} = \frac{20\sqrt{330}}{11}$

$f''(x) = + \frac{22000 \cdot 2x}{x^4} > 0$ at $x = \sqrt{22000/6}$

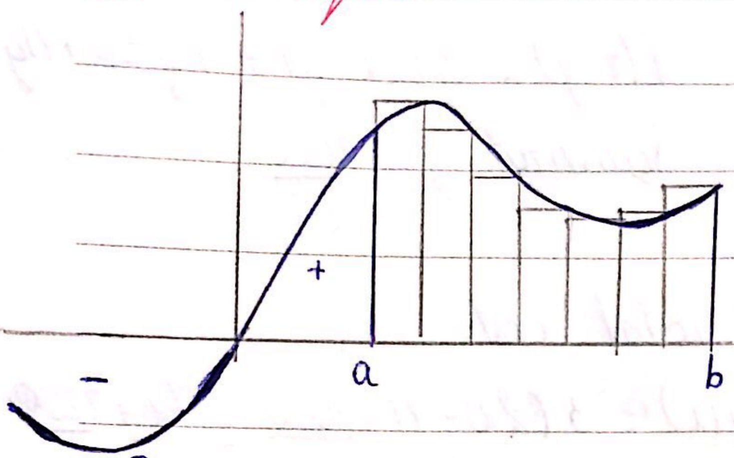
⇒ Minimum cost.

$f(\sqrt{22000/6}) = 6 \cdot \sqrt{22000/6} + \frac{22000}{\sqrt{22000/6}} = 40\sqrt{330}$

$\approx \$ 726.6$

Step 3: Answer the real life problems.
The minimum cost is $726.6, when the garden has the dimensions: length: $x = \sqrt{\frac{22000}{6}} \approx 60.6$ ft
width: $y = \frac{20\sqrt{330}}{11} \approx 33$ ft.

② Integral



- Area under the curve.
  Approximate by Rieman Sum
- Notation.
  $\int_a^b f(x)\,dx$

Can be positive, negative, $= 0$.

★ Fundamental Theorem of Calculus.

$\int_a^b f(x)\,dx = F(b) - F(a)$ (F is an antiderivative of f

★ Some basic antiderivatives to remember.

$\int x^n dx = \frac{1}{n+1} x^{n+1}$ (+c)

$\int \cos x\, dx = \sin x$

$\int \sin x\, dx = -\cos x$

$\int e^x dx = e^x$, $\int a^x dx = \frac{a^x}{\ln a}$.

$\int \frac{1}{x} dx = \ln|x|$

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$, $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a}$

$$\int \sec^2 x\,dx = \int \frac{1}{\cos^2 x}\,dx = \tan x$$

$$\int \csc^2 dx = \int \frac{1}{\sin^2 x}\,dx = -\cot x$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}$$

★ u-substitution.

EX8: $\int_1^2 \frac{4x^2}{3x^3-2}\,dx = ?$

Let $u = 3x^3 - 2 \Rightarrow du = 9x^2 dx \Rightarrow dx = \frac{du}{9x^2}$

$$\frac{4x^2}{3x^3-2}\,dx = \frac{4x^2}{u}\cdot\frac{du}{9x^2} = \frac{4\,du}{9u}$$

$x = 1 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 3.2^3 - 2 = 22.$

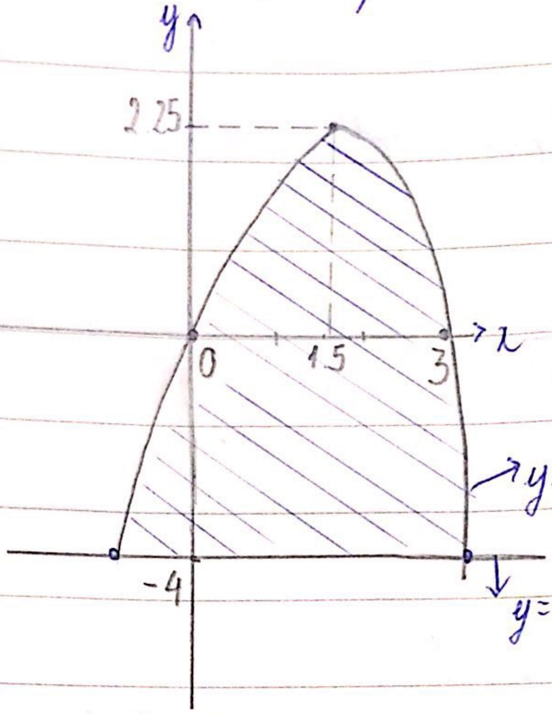
$$\int_1^2 \frac{4x^2}{3x^3-2}\,dx = \int_1^{22} \frac{4}{9u}\,du = \frac{4}{9}\ln u\Big|_1^{22} = \frac{4}{9}(\ln 22 - \ln 1)$$

$$= \frac{4}{9}\ln 22 \simeq 1.37$$

★ Areas between curves.

Ex9: Area enclosed by $y = 3x - x^2$ and $y = -4$.

• Step 0: Visualize (if possible)



• Step 1: Find the bounds of integration.

$3x - x^2 = -4$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow \left[\begin{array}{l} x = -1 \\ x = 4. \end{array}\right.$

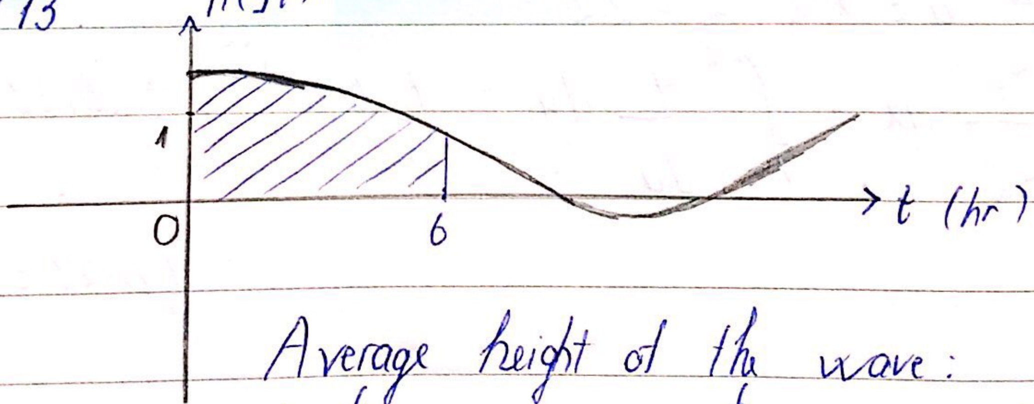
• Step 2: Which function is on the top? $y = 3x - x^2$.

• Step 3: Compute integral:

$$\int_{-1}^{4} (3x - x^2 + 4)\,dx = \frac{3}{2}x^2 - \frac{x^3}{3} + 4x \Big|_{-1}^{4} = \frac{125}{6}.$$

★ Applications of integral.

EX 13



Average height of the wave:

$$\frac{1}{6}\int_0^6 h(t)\,dt = \frac{1}{6}\int_0^6 0.6 + 0.7\cos\left(\frac{\pi}{12}t\right)dt.$$

$$\approx \frac{6.27}{6} \text{ (feet)} \approx 1.05 \text{ (feet)}$$

Ex17: $f(x) = \sin(x)$.

$T(0) = f(0) = \sin 0 = 0.$ |
$T'(0) = f'(0) = \cos(0) = 1.$ | $f'(x) = \cos x$
$T''(0) = f''(0) = -\sin(0) = 0$ | $f''(x) = -\sin x$
$T^{(3)}(0) = f^{(3)}(0) = -\cos(0) = -1.$ | $f^{(3)}(x) = -\cos x.$

$$T(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3$$

$$= 0 + 1(x) + \frac{0}{2}x^2 + \frac{(-1)}{6}x^3$$

$$= x - \frac{x^3}{6}$$

Ex18: $\int \ln x \, dx$

Let $u = x\ln x - x \Rightarrow du = (\ln x + \frac{x}{x} - 1)dx = \ln x \, dx.$

$\int \ln x \, dx = \int du = u + c = x\ln x - x + c.$