Lecture 40

Continuity and intermediate value theorem

40.1 More on continuity

Recall:

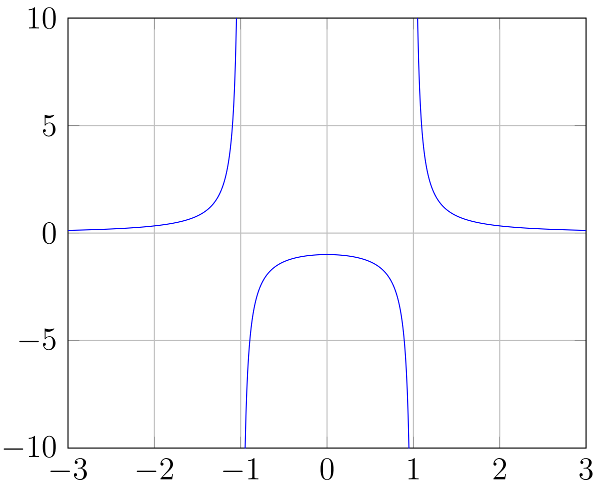
Definition 40.1.1. A function $f(x)$ is called *continuous* if it is continuous at every point that $f(x)$ is defined.

Intuition: "*A continuous function is one for which you can draw the graph of the function without ever having to lift your pencil from the paper.*"

Warning 40.1.2. This intuition fails in small ways. For example, suppose that

$$f(x) = \frac{1}{(x+1)(x-1)}.$$

Here is the graph of $f(x)$:



You can see that f is not defined at $x = 1$ and $x = -1$. So there is no way that you can draw the whole graph without lifting your pencil. But f is still a continuous function, because the value of f agrees with the limit of f at every point f is defined.

Regardless, "never have to lift your pencil" is a useful way to think about what continuity looks like. This agrees with another intuition: A continuous function has no "sudden jumps."

Example 40.1.3. As it turns out, almost every function with a "formula" that you know is continuous. Here is a list of some examples of continuous functions:

1. $f(x) = 10$ (and all other constant functions)
2. $f(x) = x$ (and all other linear functions)
3. $f(x) = 3x^3 + 4x^2 + 9$ (and all other polynomials—you can actually prove this based on the basic limit laws from last lecture)
4. $f(x) = \frac{3x^2+1}{x-3}$ (and all other functions that are quotients of polynomials—you can actually prove this based on the basic limit laws from last lecture)
5. $f(x) = |x|$ (I bet you can prove this function is continuous!)
6. $f(x) = \sin(x)$ (and all other trig functions)
7. $f(x) = \sqrt{x}$

8. $f(x) = x^p$, for any real number p, and when x is non-negative. (You should be familiar with the special cases when p is a negative integer like $p = -1$ or $p = -2$, and when p is a fraction like $p = 1/3$ or $p = 2/3$.)
9. $f(x) = e^x$
10. $f(x) = \ln(x)$

The continuity of the last five examples require some proofs that we won't go over in this class.

From now on, you may use—and are expected to know—that all the functions above are continuous.

40.2 Root law and power law from continuity

Example 40.2.1. You have now been told that $x \mapsto x^{1/n}$ is continuous. We can use the composition law to deduce the following **Root Law**: The root of the limit is the limit of the root.

That is, prove that if $\lim_{x\to a} f(x)$ exists,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

Here is the proof. Let $h(x) = x^{1/n}$. Then *because* $h(x)$ *is a continuous function,* so we can use the composition law to conclude that

$$\lim_{x\to a} h(f(x)) = h(\lim_{x\to a} f(x)). \tag{40.2.1}$$

(Line (40.2.1) is where we are using the composition law.) Now let's just plug in what $h(x)$ is to simplify both sides:

$$\lim_{x\to a} h(f(x)) = \lim_{x\to a}(f(x))^{1/n} \qquad , \qquad h(\lim_{x\to a} f(x)) = \left(\lim_{x\to a} f(x)\right)^{1/n}. \tag{40.2.2}$$

Stringing (40.2.1) and (40.2.2) together, we find:

$$\lim_{x\to a}(f(x)^{1/n}) = \left(\lim_{x\to a} f(x)\right)^{1/n}. \tag{40.2.3}$$

And now let's just remember that raising something to the $1/n$ power is the same thing as taking the nth root. So (40.2.3) becomes

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

And we're done!

Warning 40.2.2. The root law only makes sense when taking nth roots makes sense. For example, if n is even, then the law only makes sense if $\lim_{x\to a} f(x)$ is not negative.

Example 40.2.3. You have now been told that $x \mapsto x^p$ is continuous. We can use the composition law to deduce the following **Power Law**: The power of the limit is the limit of the power.

That is, prove that if $\lim_{x\to a} f(x)$ exists, then

$$\lim_{x\to a} (f(x)^p) = \left(\lim_{x\to a} f(x)\right)^p.$$

Here is the proof. Let $h(x) = x^p$. Then *because $h(x)$ is a continuous function,* we can use the composition law to conclude that

$$\lim_{x\to a} h(f(x)) = h(\lim_{x\to a} f(x)). \tag{40.2.4}$$

(Line (40.2.4) is where we are using the composition law.) Now let's just plug in what $h(x)$ is to simplify both sides:

$$\lim_{x\to a} h(f(x)) = \lim_{x\to a}(f(x))^p \qquad , \qquad h(\lim_{x\to a} f(x)) = \left(\lim_{x\to a} f(x)\right)^p. \tag{40.2.5}$$

Stringing (40.2.4) and (40.2.5) together, we find:

$$\lim_{x\to a}(f(x))^p = \left(\lim_{x\to a} f(x)\right)^p.$$

That's the power law we wanted to prove, so our proof is complete!

Warning 40.2.4. The power law only makes sense when taking pth powers makes sense. For example, if p is negative, then the law only makes sense if $\lim_{x\to a} f(x)$ is not zero.

40.3 The Intermediate Value Theorem

40.3.1 Some warm-up exercises

Exercise 40.3.1. Consider the function $f(x) = x^2 + 10$. Does this function have a root?

(Recall that a *root* is a value of x for which $f(x)$ equals zero. So, another way to rephrase the question: is there a value of x such that $x^2 + 10$ equals zero?)

Explain.

Exercise 40.3.2. Consider the polynomial function $f(x) = x^5 + 7x^4 - 22x + 19$. (This function is complicated, I know!)

Let me tell you that $f(-10)$ has the value -29,761. Also, $f(3)$ equals 763.

Based on this information, does $f(x)$ have a root?

(This question is *not* asking you to *find* a root; it's asking you whether a root *exists*.)

Explain. Can you explain in such a way where you can ignore/forget how complicated $f(x)$ looks?

In both examples, there *is* a root; this is because for the graph of a continuous function to begin with a negative height and attain a positive height, it must cross the x-axis at some point.

Here is a theorem.

Theorem 40.3.3 (Intermediate Value Theorem)**.** Let $f(x)$ be a continuous function, and choose two real numbers a and b with $a < b$[1] and assume f is defined on $[a, b]$. Then for any number N between $f(a)$ and $f(b)$, [2] there is a number c between a and b so that $f(c) = N$.

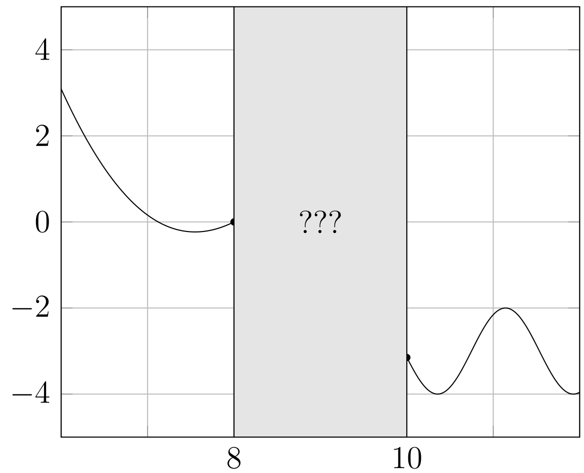
Put another way, on the way from a to b, the graph of f attains (at least) every height between $f(a)$ and $f(b)$.

Remark 40.3.4. Sometimes, we abbreviate the Intermediate Value Theorem by "IVT" (especially when we are running out of time on exams or quizzes).

Example 40.3.5. Here is a graph of a function $f(x)$ that your friend began to make, then stopped part-way:

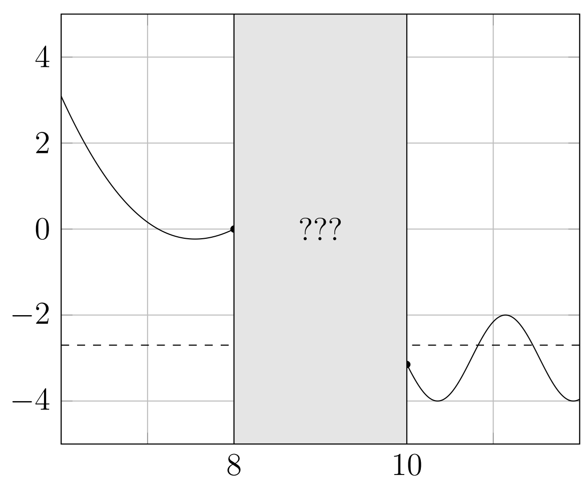
[1] You should imagine these numbers to be on the x-axis.

[2] You should imagine N, $f(a)$, and $f(b)$ to be on the y-axis



So you have no idea what $f(x)$ looks like in the region between 8 and 10. However, you do know that $f(8) = 0$ and $f(10) = -3$. Therefore, *if $f(x)$ is continuous*, then the Intermediate Value Theorem tells you that $f(x)$ must hit (at least) every number between 0 and -3, at least once.[3]

For example, -2.7 is a number between 0 and -3. So, though you *do not know where*, you do know that $f(x)$ must equal -2.7 at *some value* of x between 8 and 10. [4] Here is a pictorial way to think about it:



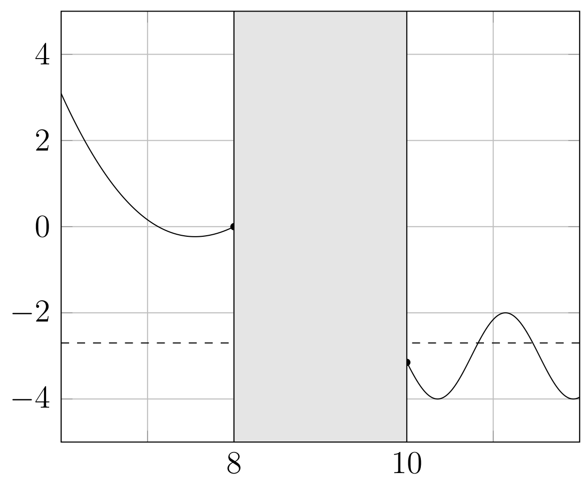
[3]In this example, $a = 8$ and $b = 10$.

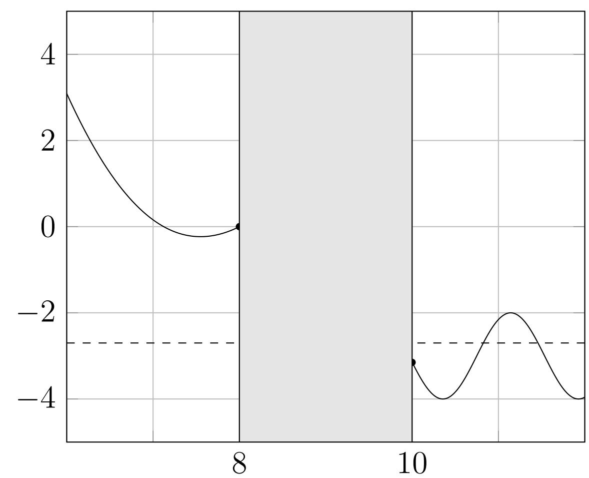
[4]In terms of the letters used in Theorem 40.3.3, $N = -2.7$. And c is the *some value* between 8 and 10.

We have drawn, in dashes, the line at height -2.7. Because $f(x)$ is continuous, to get from height 0 to height -3, the graph of $f(x)$ *must* cross over this line at some point in the grey region. We don't know where $f(x)$ crosses the line, but it does so *somewhere* between $x = 8$ and $x = 10$.

Remark 40.3.6. Note that, in Example 40.3.5, the graph of $f(x)$ crosses over the line of height -2.7 *outside* the grey region as well. That's all well and good, but the intermediate value theorem only guarantees something about the *grey region*—i.e., about the region between a and b.

Remark 40.3.7. Here are some examples of continuous functions that could fill in the grey region from Example 40.3.5:





Note that $f(x)$ may attain N at *more than one value of* c. (You can see this graphically in the lefthand example: The graph of $f(x)$ crosses the horizontal line of height $N = -2.7$ three times.)

Note that $f(x)$ *does not need to stay inbetween* $f(a)$ *and* $f(b)$. (You can see this on the righthand example.) That is, even if $a < c < b$, it need *not* be true that $f(c)$ is between $f(a)$ and $f(b)$.

Exercise 40.3.8. Do Exercise 40.3.2 again, using the IVT. Make sure you know what the values of a, b, and N are.

Do you know the value of c?

40.4 Intermediate value theorem on a closed interval

Recall that a *closed* interval is an interval of the form

$$[a, b]$$

with $a < b$. For example, $[2, 7]$ is the interval of all numbers between 2 and 7, *including* 2 and 7.

An *open* interval is an interval of the form

$$(a, b)$$

with $a < b$. For example, $(2, 7)$ is the interval of all numbers between 2 and 7, *not including* 2 and 7.

If a function $f(x)$ is defined only on a closed interval $[a, b]$, it's not obvious what we mean for f to be continuous—mainly because we can only define a one-sided limit (and not a limit) at a and b. But we take what we can get:

Definition 40.4.1. If a function $f(x)$ is defined only on a closed interval $[a, b]$, we say that f is *continuous at* a if

1. The righthand limit $\lim_{x \to a^+} f(x)$ exists, and
2. $\lim_{x \to a^+} f(x) = f(a)$.

Likewise, we say that f is *continuous at* b if

1. The lefthand limit $\lim_{x \to b^-} f(x)$ exists, and
2. $\lim_{x \to b^-} f(x) = f(b)$.

We say that f is continuous if it is continuous at every point of $[a, b]$. [5]

Theorem 40.4.2. The intermediate value theorem holds for continuous functions defined on a closed interval.

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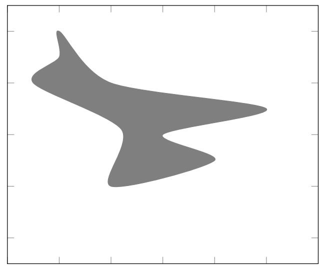
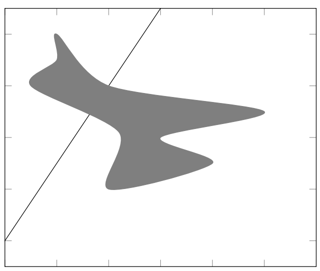
[5] Note that for any element c inside of (a, b)—that is, for any c with $a < c < b$—we know what it means for $f(x)$ to be continuous at c, because we know how to define the limit of f at c.

40.5 A fun exercise: Wonky pizza

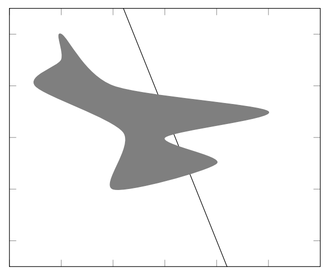
Here is a picture of a wonky-shaped pizza. (And yes, it's gray; not the most tasty-looking thing, is it?)



Your boss wants you to cut this pizza in half, using *one*, linear cut. For example,



and



are two cuts you're allowed to make. Notice that the resulting pizza can have more than just two pieces (as seen on the righthand cut). All that your boss wants is that all the pizza on one side of the cut, has the same area as all the pizza on the other side of the cut.

Exercise 40.5.1. Using the Intermediate Value Theorem, convince yourself that for *any* slope m you choose, you can make a cut of slope m such that you divide the pizza into equal halves (just as your boss requires).

Does the theorem tell you *where* to cut the pizza?