Lecture 39

L'Hôpital's rule and curve-sketching

39.1 L'Hôpital's rule

Exercise 39.1.1. Remember that

$$\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1.$$

For no good reason, let's take the derivative of the top and bottom functions, and then take the limit:

$$\lim_{x\to 0^+} \frac{(\sin(x))'}{(x)'}.$$

What answer do you get?

Exercise 39.1.2. Compute the limit

$$\lim_{x\to\infty} \frac{2x+3}{5x-7}.$$

Let's try taking the derivative of the top and bottom function first, and then take the limit. That is, compute

$$\lim_{x\to\infty} \frac{(2x+3)'}{(5x-7)'}.$$

How do your answers compare?

Exercise 39.1.3. Compute the limits

$$\lim_{x\to\infty} \frac{x^2}{1/x} \qquad \text{and} \qquad \lim_{x\to\infty} \frac{(x^2)'}{(1/x)'}.$$

How do your answers compare?

The first two exercises were promising, but the last one showed that this trick doesn't always work. Here is a theorem that you may use freely; we won't prove it in this class:

Theorem 39.1.4 (L'Hôpital's Rule)**.** Let f and g be functions. If

1. $\lim f(x) = \infty$ and $\lim g(x) = \infty$, or if
2. $\lim f(x) = 0$ and $\lim g(x) = 0$,

then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

provided the righthand side exists.

In words, L'Hôpital's Rule says that: If the limits of f and g are both some sort of infinity, or both zero, then the limit of the fraction may be computed by first taking the derivatives of f and g (so long as the limit of the derivatives exist).

Remark 39.1.5. Some textbooks use instead the condition $\lim |f(x)| = \infty$ (and likewise for g), which can seem a little confusing. It turns out this condition is identical to "$\lim f(x) = \infty$ or $\lim f(x) = -\infty$" and likewise for g. (In general, it is very rare for a function to satisfy $\lim |f(x)| = \infty$ without satisfying $\lim f(x) = \infty$ or $\lim f(x) = -\infty$. In fact, such a scenario is impossible if f is continuous and defined for all large values of x. And secretly, we are assuming that f is defined for all large values of x when we compute limits of f as x approaches infinity.)

In either case, these cases all follow from what we've stated above. For example, by the scaling law, $\lim f(x) = -\lim(-f(x))$, so we can always convert a limit equaling $-\infty$ to one equaling ∞.

Warning 39.1.6. The hypothesis of L'Hôpital's Rule is important! (The limits of the denominator and numerator must both agree.) You saw in Exercise 39.1.3 an example where the numerator and denominator had different limits; as a result, the limit of the fraction after taking the derviatives was *different* from the limit of the fraction.

It may also be that the limit of f/g exists, but the limit of f'/g' doesn't exist. Example: $f(x) = x + \cos x$ and $g(x) = x$ and the limit as $x \to \infty$.

Remark 39.1.7. The limits in the statement of L'Hôpital's Rule have no subscripts. This is because I am being lazy. To be explicit: If all the limits are taken at the same point, then the theorem holds.

For example, if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^+} g(x)$ both equal zero, you can apply L'Hôpital's Rule:

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

This works for one-sided limits from the left, and for limits at $\pm\infty$.

Remark 39.1.8. As you may have guessed, "L'Hôpital" is a French name. It is pronounced (roughly) "Lo-pee-tahl." You may not be used to the ô; that is, to the little "hat" on top of the *o*. This symbol is called a *circumflex,* and in French, it is often used when a word *used* to have an *s* right after the circumflex. So for example, in the past, the word "L'Hôpital" would have been spelled "L'Hospital." Yes, that's right; this person's name literally translates to "The Hospital."

Exercise 39.1.9. Evaluate the following limits. Some may involve L'Hôpital's rule; other may not. When you use L'Hôpital's rule, say why you know you can use it (based on the hypotheses of the theorem above).

(a) $\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)}$

(b) $\lim_{x\to\infty} \frac{x}{x^2-1}$

(c) $\lim_{x\to 1^+} \frac{x}{x^2-1}$

(d) $\lim_{x\to-\infty} \frac{1}{2x+3}$

(e) $\lim_{x\to 0^+} x\ln x$

(f) $\lim_{x\to 0^-} \frac{x}{\sin(x)}$

(g) $\lim_{x\to\infty} xe^x$

(h) $\lim_{x\to-\infty} xe^x$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

39.2 For next time

You should be able to compute all the limits above (and limits similar to them).

(Solutions are on next page if necessary.)

Solutions

(a) $\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)}$

Evaluating the limit in the numerator and denominator yields

$$\frac{\lim_{x\to(\pi/2)^+}(x-\pi/2)\sin(x)}{\lim_{x\to(\pi/2)^+}\cos(x)} = \frac{(\pi/2-\pi/2)\cdot 1}{0} \tag{39.2.1}$$

This is 0/0, so we can use L'Hôpital's rule.

$$\lim_{x\to(\pi/2)^+} \frac{(x-\pi/2)\sin(x)}{\cos(x)} = \lim_{x\to(\pi/2)^+} \frac{((x-\pi/2)\sin(x))'}{(\cos(x))'} \tag{39.2.2}$$

$$= \lim_{x\to(\pi/2)^+} \frac{\sin(x)+(x-\pi/2)\cos(x)}{-\sin(x)} \tag{39.2.3}$$

$$= \lim_{x\to(\pi/2)^+} \frac{1+0}{-1} \tag{39.2.4}$$

$$= -1. \tag{39.2.5}$$

(b) $\lim_{x\to\infty} \frac{x}{x^2-1}$

Evaluating limits in the numerator and denominator, we obtain ∞/∞, so we can use L'Hôpital's rule.

$$\lim_{x\to\infty} \frac{x}{x^2-1} = \lim_{x\to\infty} \frac{(x)'}{(x^2-1)'} \tag{39.2.6}$$

$$= \lim_{x\to\infty} \frac{1}{2x} \tag{39.2.7}$$

$$= 0. \tag{39.2.8}$$

You also could have solved the original limit without L'Hôpital's Rule: Just divide top and bottom by x.

(c) $\lim_{x\to 1^+} \frac{x}{x^2-1}$

We *cannot* use L'Hôpital's Rule here because, when evaluating the limits of the numerator and denominator, we arrive at 1/0. This is not 0/0 nor ∞/∞.

But we can still divide top and bottom by x. Then

$$\lim_{x\to 1^+} \frac{x}{x^2-1} = \lim_{x\to 1^+} \frac{x}{x^2-1} \cdot \frac{1/x}{1/x} \tag{39.2.9}$$
$$= \lim_{x\to 1^+} \frac{1}{x-\frac{1}{x}} \tag{39.2.10}$$
$$= \frac{\lim_{x\to 1^+} 1}{\lim_{x\to 1^+} x-\frac{1}{x}} \tag{39.2.11}$$
$$= \frac{1}{\lim_{x\to 1^+} x-\frac{1}{x}}. \tag{39.2.12}$$

When $x > 1$, we know that $x - 1/x$ is positive. So the denominator approaches 0 from the right.

$$= \frac{1}{0^+} = \infty.$$

Here is another way you could have computed this limit. Note that $(x^2 - 1) = (x + 1)(x - 1)$, and we know that $(x - 1)$ is the factor that is causing the denominator to become 0 in the limit. So let's rewrite things in a way we can try to factor out an $(x - 1)$ from the *numerator*, too:

$$\lim_{x\to 1^+} \frac{x}{x^2-1} = \lim_{x\to 1^+} \frac{x}{(x-1)(x+1)} \tag{39.2.13}$$
$$= \lim_{x\to 1^+} \frac{x-1+1}{(x-1)(x+1)} \tag{39.2.14}$$
$$= \lim_{x\to 1^+} \frac{x-1}{(x-1)(x+1)} + \lim_{x\to 1^+} \frac{1}{(x-1)(x+1)} \tag{39.2.15}$$
$$= \lim_{x\to 1^+} \frac{1}{x+1} + \lim_{x\to 1^+} \frac{1}{(x-1)(x+1)} \tag{39.2.16}$$
$$= \lim_{x\to 1^+} \frac{1}{1} + \lim_{x\to 1^+} \frac{1}{x^2-1} \tag{39.2.17}$$
$$= 1 + \lim_{x\to 1^+} \frac{1}{x^2-1}. \tag{39.2.18}$$

Now note that $x^2 - 1$ approaches 0 from the right when $x \to 1^+$, because if $x > 1$, then $x^2 > 1$. So this limit becomes

$$= 1 + \frac{1}{0^+} = 1 + \infty = \infty$$

just as before.

(d) $\lim_{x\to-\infty} \frac{1}{2x+3}$ We don't need L'Hôpital's Rule: We see this limit is $1/-\infty = 0$. Note that we couldn't have used L'Hôpital's Rule anyway.

(e) $\lim_{x\to 0^+} x \ln x$

Let's rewrite this limit:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \tag{39.2.19}$$

$$= -\lim_{x\to 0^+} \frac{-\ln x}{1/x}. \tag{39.2.20}$$

You don't really need to insert the minus sign, but I did so to see that $\lim_{x\to 0^+} -\ln x = \infty$ and $\lim_{x\to 0^+} = \infty$; this shows we can apply L'Hôpital's Rule. Applying said rule, we find

$$-\lim_{x\to 0^+} \frac{-\ln x}{1/x} = -\lim_{x\to 0^+} \frac{(-\ln x)'}{(1/x)'} \tag{39.2.21}$$

$$= -\lim_{x\to 0^+} \frac{-1/x}{-1/x^2} \tag{39.2.22}$$

$$= -\lim_{x\to 0^+} \frac{1/x}{1/x^2} \tag{39.2.23}$$

$$= -\lim_{x\to 0^+} \frac{1/x}{1/x^2} \cdot \frac{x^2}{x^2} \tag{39.2.24}$$

$$= -\lim_{x\to 0^+} \frac{x}{1} \tag{39.2.25}$$

$$= -\frac{0}{1} \tag{39.2.26}$$

$$= 0. \tag{39.2.27}$$

(f) $\lim_{x\to 0^-} \frac{x}{\sin(x)}$

We already know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, so we can use that to say

$$\lim_{x\to 0^-} \frac{x}{\sin(x)} = \lim_{x\to 0^-} \frac{1}{\frac{x}{\sin(x)}} \tag{39.2.28}$$

$$= \frac{1}{\lim_{x\to 0^-} \frac{x}{\sin(x)}} \tag{39.2.29}$$

$$= \frac{1}{1} \tag{39.2.30}$$

$$= 1. \tag{39.2.31}$$

You could also use L'Hôpital's Rule to arrive at

$$\lim_{x\to 0^+} \frac{1}{\cos(x)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$$

(g) $\lim_{x\to\infty} xe^x$ You don't need L'Hôpital's Rule here; we plainly see that the limit is given by $\infty \cdot \infty = \infty$.

(h) $\lim_{x\to-\infty} xe^x$ This gets trickier, because we find (taking naive limits)

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} x \lim_{x\to-\infty} e^x = (-\infty)\cdot(0)$$

which is undefined. So let's rewrite this limit as a fraction:

$$\lim_{x\to-\infty} xe^x == \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{39.2.32}$$

$$= \frac{\lim_{x\to-\infty} x}{\lim_{x\to-\infty} e^x} \tag{39.2.33}$$

$$= -\frac{\lim_{x\to-\infty} -x}{\lim_{x\to-\infty} e^x} \tag{39.2.34}$$

$$= -\frac{\infty}{\infty} \tag{39.2.35}$$

which means we can use L'Hôpital's Rule on this limit. We find:

$$\lim_{x\to-\infty} xe^x == \lim_{x\to-\infty} \frac{x}{e^{-x}} \tag{39.2.36}$$

$$= \frac{\lim_{x\to-\infty}(x)'}{\lim_{x\to-\infty}(e^x)'} \tag{39.2.37}$$

$$= \frac{\lim_{x\to-\infty} 1}{\lim_{x\to-\infty} e^x} \tag{39.2.38}$$

$$= \frac{1}{\lim_{x\to-\infty} e^x} \tag{39.2.39}$$

$$= \frac{1}{0^+} \tag{39.2.40}$$

$$= \infty. \tag{39.2.41}$$

$$\tag{39.2.42}$$

(i) $\lim_{x\to\infty} \frac{5^x}{x^2}$

Evaluating the numerator and denominator limits, we obtain ∞/∞, so we can use L'Hôpital's Rule. Then we end up with

$$\lim_{x\to\infty} \frac{\ln 55^x}{2x}.$$

Taking limits of top and bottom, again we find ∞/∞. So we can use L'Hôpital's Rule *again*. Then we find

$$\lim_{x\to\infty} \frac{(\ln 5)^2 5^x}{2}.$$

The limit of this expression is clearly ∞.

(j) $\lim_{x\to\infty} \frac{5^x}{x^3}$

This problem is the same work as above, but you use L'Hôpital's Rule three times.

Curve-sketching

Without using a graphing calculator, let's visualize the function

$$f(x) = \frac{x^2+1}{x^2-2}$$

using tools of calculus!

39.3 Asymptotes

First, let's find the vertical and horizontal asymptotes. Remember, you do this by computing

1. The limits at $\pm\infty$ (to find horizontal asymptotes), and
2. The limits where f looks undefined (there are vertical asymptotes if this limit is $\pm\infty$).

(1) You can compute that the horizontal asymptote is 1, and that f approaches 1 near both ∞ and $-\infty$. Here's the computation for the limit at $-\infty$; I'll leave the other limit to you!

$$\lim_{x\to-\infty}\frac{x^2+1}{x^2-2} = \lim_{x\to-\infty}\frac{x^2+1}{x^2-2} \tag{39.3.1}$$

$$= \lim_{x\to-\infty}\frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}} \tag{39.3.2}$$

$$= \frac{\lim_{x\to-\infty} 1+\frac{1}{x^2}}{\lim_{x\to-\infty} 1-\frac{2}{x^2}} \tag{39.3.3}$$

$$= \frac{1+0}{1-0} \tag{39.3.4}$$

$$= 1. \tag{39.3.5}$$

(You *do* need to compute both limits, because there may be horizontal asymptotes with different heights.)

(2) f potentially has asymptotes where the denominator is zero—that is, when $x = \pm\sqrt{2}$. There are *four* one-sided limits to compute here. I will compute one for you, and tell you the answer for the rest:

$$\lim_{x\to-\sqrt{2}^-} \frac{x^2+1}{x^2-2} = \lim_{x\to-\sqrt{2}^-} \frac{x^2+1}{x^2-2} \tag{39.3.6}$$

$$= \frac{\lim_{x\to-\sqrt{2}^-} x^2+1}{\lim_{x\to-\sqrt{2}^-} x^2-2} \tag{39.3.7}$$

$$= \frac{2+1}{0^+} \tag{39.3.8}$$

$$= \frac{3}{0^+} \tag{39.3.9}$$

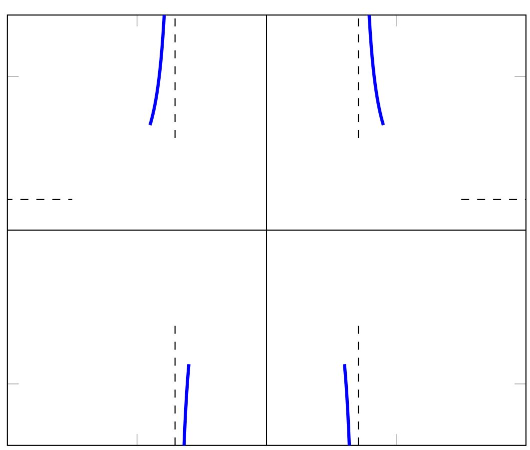
$$= \infty. \tag{39.3.10}$$

The thing that requires most explanation is probably how we got to line (39.3.8). As x approaches $-\sqrt{2}$ from the left, x^2 approaches 2 *from the right*; that is, x^2 is shrinking toward 2. Thus, $x^2 - 2$ is approaching 0 from the right. This is why the denominator becomes 0^+.

The three other one-sided limits can be computed to be

$$\lim_{x\to-\sqrt{2}^+} \frac{x^2+1}{x^2-2} = -\infty, \qquad \lim_{x\to\sqrt{2}^-} \frac{x^2+1}{x^2-2} = -\infty, \qquad \lim_{x\to\sqrt{2}^+} \frac{x^2+1}{x^2-2} = \infty.$$

Based purely on these computations, we can begin to visualize the graph of $f(x)$:



Tbe dashed lines show where the asymptotes are; the thick blue lines are the beginnings of the graph. Note that I haven't yet drawn the parts of the graph near horizontal $\pm\infty$; this is because while I know the asymptotes, I don't know how f approaches the asymptotes (for example, f could oscillate close to the asymptote, or approach from above, or from below).

39.4 Concavity

Now I'd recommend computing the concavity of the function to get a good feel for shape.

Remember, concavity is dictated by whether the second derivative is positive or negative. So let's compute the second derivative.

First, we compute f':

$$\left(\frac{x^2+1}{x^2-2}\right)' = \frac{(x^2+1)'(x^2-2)-(x^2-2)'(x^2+1)}{(x^2-2)^2} \tag{39.4.1}$$

$$= \frac{(2x)(x^2-2)-(2x)(x^2+1)}{(x^2-2)^2} \tag{39.4.2}$$

$$= \frac{(2x)(x^2-2-(x^2+1))}{(x^2-2)^2} \tag{39.4.3}$$

$$= \frac{(2x)(-3)}{(x^2-2)^2} \tag{39.4.4}$$

$$= \frac{-6x}{(x^2-2)^2} \tag{39.4.5}$$

The second derivative is computed as follows:

$$\left(\frac{x^2+1}{x^2-2}\right)'' = \left(\frac{-6x}{(x^2-2)^2}\right)' \tag{39.4.6}$$

$$= \frac{(-6x)'(x^2-2)^2 - (-6x)((x^2-2)^2)'}{(x^2-2)^4} \tag{39.4.7}$$

$$= \frac{(-6)(x^2-2)^2 - (-6x)(x^2-2)(2x)}{(x^2-2)^4} \tag{39.4.8}$$

$$= \frac{(-6)(x^2-2)[(x^2-2)-(x)(2x)]}{(x^2-2)^4} \tag{39.4.9}$$

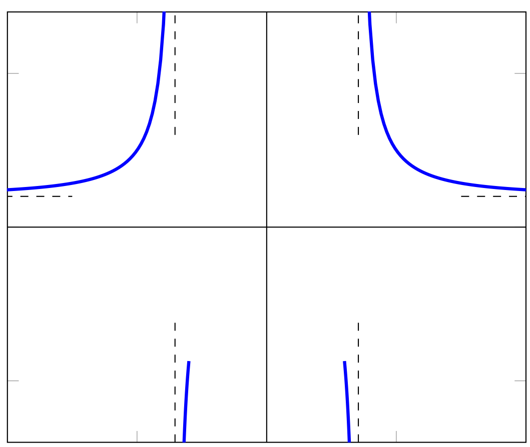
$$= \frac{(-6)(x^2-2)(-x^2-2)}{(x^2-2)^4} \tag{39.4.10}$$

$$= \frac{6(x^2-2)(x^2+2)}{(x^2-2)^4} \tag{39.4.11}$$

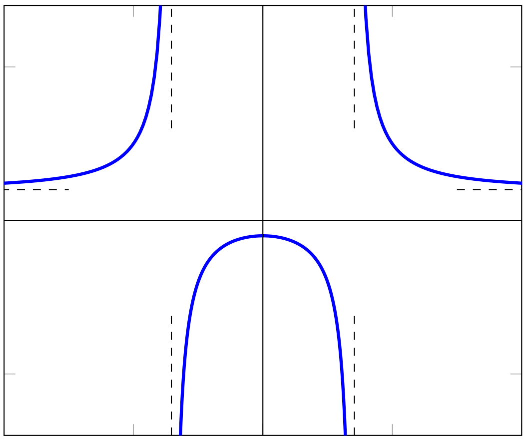
When is this fraction positive, and when is it negative?

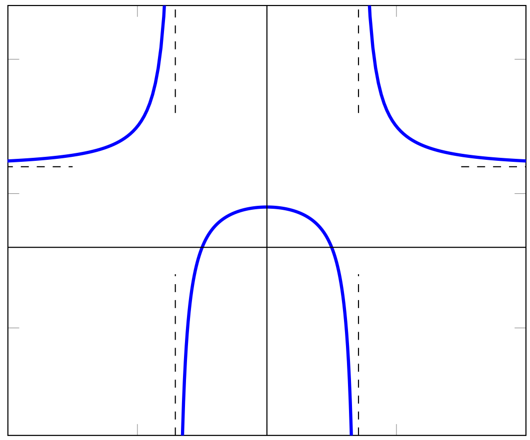
The denominator, $(x^2-2)^4$, is always positive, so we can focus on the numerator, $6(x^2-2)(x^2+2)$.

(i) We see that x^2+2 is always positive, while x^2-2 is positive whenever $x^2 > 2$. That is, whenever $|x| > \sqrt{2}$. At this point, we know that the graph of the function must be concave up when $|x| > \sqrt{2}$. So we can begin to draw this portion of the graph:



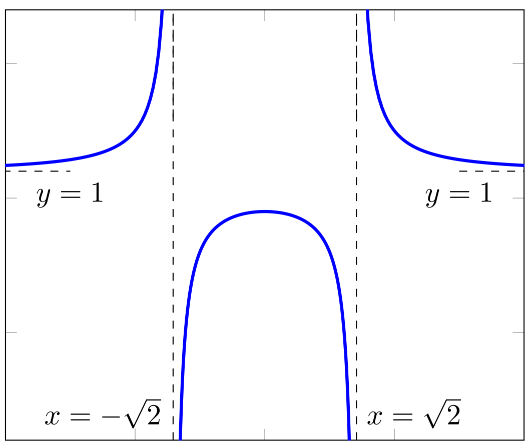
(ii) $6(x^2-2)(x^2+2)$ is negative precisely when $|x| < \sqrt{2}$, by the same reasoning as before; so we know that the function will look concave down in this region. So we can conclude the graph probably looks like one of the following:





At this point, there is still ambiguity in what the graph actually looks like. Where is the local maximum in the middle? At what x-coordinate? And at what y-coordinate?

But, depending on the kind of information you're looking for, you might be satisfied with a vague sketch as follows:



39.5 If you want more

If you want more information, or are asked for more information, you can make a more accurate sketch by finding out things such as:

- Identifying critical points.

- Finding the y- and x-intercepts.
- Labeling inflection points (in our case, we had none).

39.6 Summary and motivation

Let me emphasize one thing: **Your computations by hand are often more reliable than what graphing calculators will show you.** Being able to identify the critical points, the asymptotes, et cetera, can even tell you what frame you should use to look at a graph (e.g., what x values and what y values should your window hold?).

Here is a **summary of curve-sketching**: Identify the asymptotes, identify the concavity of the important regions, and then collect more information if you need (critical points, intercepts, et cetera).

Example 39.6.1. You are told that a function f has the following properties:

(a) $\lim_{x\to-\infty} f(x) = 3$.

(b) $\lim_{x\to\infty} f(x) = 2$.

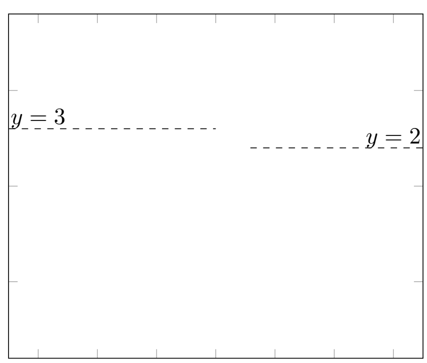
(c) f is continuous and defined everywhere.

(d) $f''(x)$ is positive when x is between -1 and 5

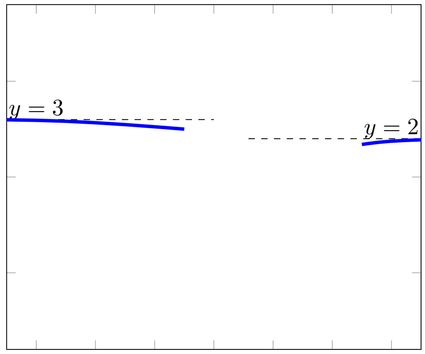
(e) $f''(x)$ is negative when $x < -1$ and when $x > 5$.

Sketch the graph.

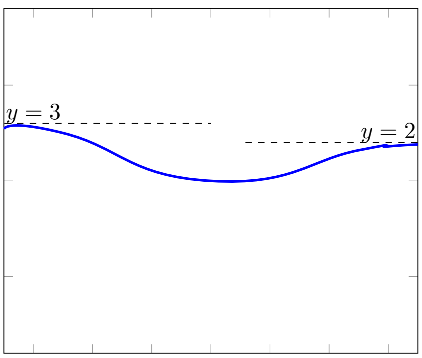
Solution: Based on (a) and (b), we can first draw the horizontal asymptotes, though we don't know how f approaches these asymptotes yet.



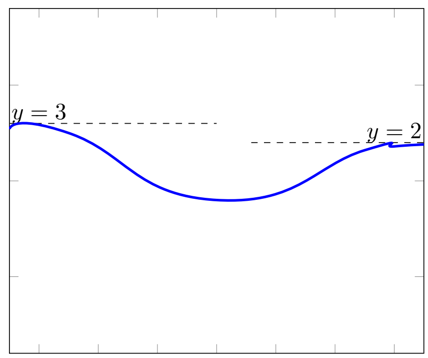
We know there are no vertical asymptotes by (c). By (e), we know that the function looks concave down outside of the interval $[-1, 5]$, so we can begin to draw as follows:



By (d), the rest of the function is concave up. So we sketch a "bowl up" shape:



or



(With the information given, it is impossible to draw the graph of f with complete accuracy, but you see that you get a "feel" for what it looks like!)

39.7 For next time

For next time, I expect you to able to sketch the following graphs

(a) $f(x) = \frac{1}{x^2-2}$ (explaining *why* the sketch looks the way it does)

(b) $f(x) = \frac{1}{e^x-3}$ (explaining *why* the sketch looks the way it does)

(c) A continuous function f satisfying the following properties:

  (a) $\lim_{x\to\infty} f(x) = 5$

(b) $\lim_{x\to-\infty} f(x) = -5$
(c) $\lim_{x\to 2^+} f(x) = \infty$
(d) $\lim_{x\to 2^-} f(x) = \infty$
(e) $f''(x) < 0$ when x is less than -10,
(f) $f''(x) > 0$ when x is between -10 and 2,
(g) $f''(x) > 0$ when x is larger than 2.

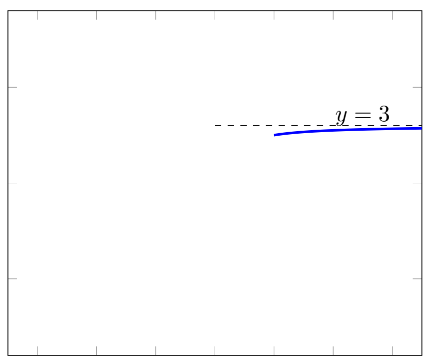
39.8 Appendix: Concavity near ∞

As you get to sketching many curves, it's natural to ask the following question: If I know

- $\lim_{x\to\infty} f(x) = 3$ and
- $f''(x) < 0$,

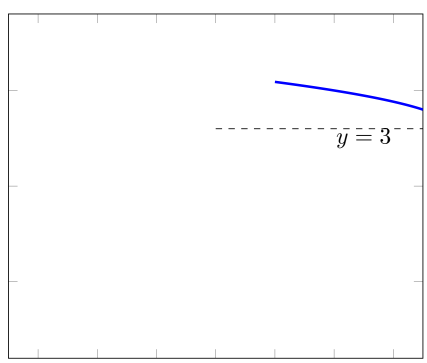
why do I know that f has to look like

DRAWING I:



near ∞? For example, why couldn't f look like the following?

DRAWING II:



In this appendix, I claim: **Drawing II can never happen** while f satisfies the two properties above. That is, if f is concave down and has a horizontal asymptote, f must approach that asymptote from *underneath* the asymptote, not from above.

Here is a great place for *proof.* Let's try to put into mathematical language what picture you're drawing in Drawing II: You seem to be drawing a function with

1. $f'' < 0$ for x larger than (for example) 5, and

2. $f' < 0$ for x larger than 5.

I want to emphasize that the role of 5 could be swapped with any number; so let's just call that number a from now on.

I claim the following:

Proposition 39.8.1. Suppose that f is a function such that, for some real number a, we have

1. $f''(x) < 0$ for all $x > a$, and

2. $f'(x) < 0$ for all $x > a$.

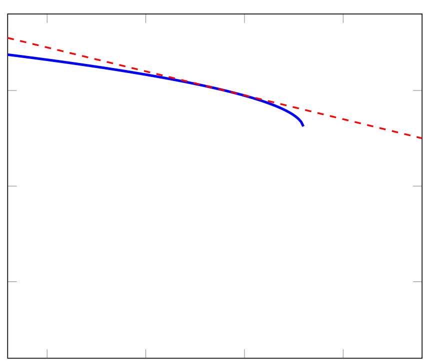
Then $\lim_{x\to\infty} f(x) = -\infty$. In particular, f does not have a horizontal asymptote as $x \to \infty$.

Proof. Let b be any number bigger than a, and let's let $M = f'(b)$. (That is, M is the slope of the tangent line to f at b.)

Then we can define another function called

$$g(x) = M(x - b) + f(b).$$

The graph of g is a line—a line with slope M, and which passes through the point $(b, f(b))$.



Above, the graph of g has been represented as a dashed red line, and the graph of f as a solid blue curve. Note g has negative slope because of hypothesis 2. of the Proposition.

Now, because $f''(x) < 0$, we know that $f'(x)$ will be less than $f'(b)$ for all $x > b$. This is a consequence of the mean value theorem! It's because you can think of $h = f'(x)$ as a function; and when $h'(x) < 0$ for all $x > a$ (because of Hypothesis 1), we know that h is decreasing in value for all $x > a$, so $h(x)$ will be less than $h(b)$ whenever $x > b$.

Now, because $f'(x) < f'(b) = g'(x)$ for all $x > b$, we see that $f(x) < g(x)$ for all $x > b$. (This is an application of the mean value theorem again.)

So if $f(x) < g(x)$ for all $x > b$, we conclude that

$$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x).$$

But $g(x) = M \cdot (x - b) + f(b)$. So we see that

$$\begin{aligned}
\lim_{x\to\infty} g(x) &= M \cdot \left(\lim_{x\to\infty} (x-b)\right) + f(b) && (39.8.1)\\
&= M \cdot (\infty - b) + f(b) && (39.8.2)\\
&= M \cdot (\infty) + f(b) && (39.8.3)\\
&= -\infty + f(b) && (39.8.4)\\
&= -\infty. && (39.8.5)
\end{aligned}$$

(Note that $M \cdot \infty = -\infty$ because $M < 0$.)

Putting everything together, we see

$$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x) = -\infty$$

so $\lim_{x\to\infty} f(x) = -\infty$. □