Lecture 33

The squeeze theorem

Today we're going to see something called the "squeeze theorem." It allows us to compute limits we couldn't compute before.

Theorem 33.0.1 (The squeeze theorem)**.** Let f, g, h be functions, and let a be a number. Suppose that for every x near a (but not necessary for x equal to a), we have inequalities

$$f(x) \leq g(x) \leq h(x).$$

(In words, the graph of f is always under the graph of g, and the graph of g is always under the graph of h—near but not at a.)

Then, if the limits of f and h at a exist and if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$$

then the limit of g at a exists, and in fact

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x).$$

Figure 33.1 is a picture representing the algebra above. Drawn with thick lines are the graphs of f and h. Notice that f is always below h near a, but that both f and h convergeto the same point (indicated by the circle) as x approaches a.

Finally, note that the graph of g is always between the graph of f and h near a. The picture is meant to show that the limit of $g(x)$ as x approaches a is indeed "squeezed" to be equal to the limit of f and h at a.

Example 33.0.2. Let $g(x) = \frac{\sin(x)}{x}$. Note g is not defined at $x = 0$. Next class, we will see an example of two functions f and h for which

$$f(x) \leq \frac{\sin(x)}{x} \leq h(x)$$

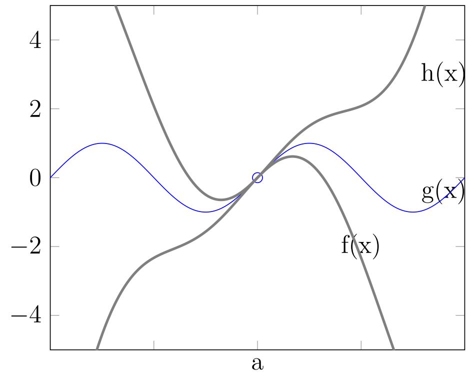

Figure 33.1: The squeeze theorem

whenever x is near 0. Moreover, we will see that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 1.$$

By the squeeze theorem, we will conclude that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

This was a fact we learned in week two of class! (We used it to prove that $(\sin(x))' = \cos(x)$.

Today, I want to show you the proof of the squeeze theorem. It will give you an idea of how the ϵ-δ definition of limit is used to prove powerful statements (like the squeeze theorem).

Proof of the squeeze theorem. Fix some $\epsilon > 0$.

Let L be the limit of $f(x)$ and $h(x)$ as x approaches a; this means that, for the ϵ we just fixed, there exists a δ_f and a δ_h so that

$$|x-a| < \delta_f \implies |f(x) - L) < \epsilon \qquad \text{and} \qquad |x-a| < \delta_h \implies |h(x) - L| < \epsilon.$$

I claim that if δ is any positive number less than δ_f and less than δ_h, then

$$|x-a| < \delta \implies |g(x) - L| < \epsilon.$$

(If this claim is true, then we've proven that the limit of g as x approaches a is indeed L.)

So let's see why the claim is true. The proof is a little bit wonky, so stick with me.

First, we are told that $f(x) \leq g(x) \leq h(x)$ for every x near a. So, subtracting L from all sides, we know that

$$f(x) - L \leq g(x) - L \leq h(x) - L. \tag{33.0.1}$$

Next, regardless of x, we know that either $g(x) - L \leq 0$, or $g(x) - L \geq 0$. ($g(x) - L$ could equal 0 too, but this won't matter.)

- If $g(x) - L \geq 0$, then $|g(x) - L| = g(x) - L$ by definition of absolute value. And by (33.0.1), we see then that $h(x) - L \geq 0$, too, so $|h(x) - L| = h(x) - L$. This means

  $$|g(x) - L| = g(x) - L \leq h(x) - L = |h(x) - L|,$$

  so

  $$|g(x) - L| \leq |h(x) - L|.$$

- If $g(x) - L \leq 0$, then $|g(x) - L| = -(g(x) - L)$ by definition of absolute value. And by (33.0.1), we see then that $f(x) - L \leq 0$, too, so $|f(x) - L| = -(f(x) - L)$. This means

  $$|g(x) - L| = -(g(x) - L) \leq -(f(x) - L) = |f(x) - L|,$$

  so

  $$|g(x) - L| \leq |f(x) - L|.$$

So, regardless of whether $g(x) - L$ is positive or negative, we see that $|g(x) - L|$ is always less than either $|f(x) - L|$ or $|h(x) - L|$.

Well, if δ is any number less than both δ_f and δ_h, we see that $|f(x) - L|$ and $|h(x) - L|$ are both less than ϵ. Therefore, $|g(x) - L|$ is also less than ϵ.

This completes the proof. $\square$

This proof was long. And it will take most of us a few days to understand it, especially if this is the first time we've ever seen such an abstract proof. But this is similar to understanding the engine of a car. Understanding *why* the engine of a car works would take us a few days (at least). But the miracle of science is that, once we understand, we can be confident that the engine works. Moreover, if we ever encounter a situation where the engine doesn't work, we have a hope of understanding why the engine may have stopped working.

In this analogy, the definition of limit (using ϵ-δ) is the science behind the engine. The above proof shows how to use this science to know that a very intuitive fact (like the squeeze theorem) *must* be true.

33.1 What are you supposed to take away from this?

Based on this lecture, you should know (i) the statement of the squeeze theorem. For example, what are its hypotheses (e.g., when can you invoke it)? And what are its conclusions (if you invoke it, what truth can you utilize)? You will not see too many problems in this course that ask you to use the squeeze theorem, but the squeeze theorem is very important—it is the theorem that allows us to compute the limit of $\sin(x)/x$ as x goes to zero. (Next lecture.)

The other thing you should gain is (ii) the experience of seeing a proof, reading a proof, and trying to understand the proof. Mathematics is dense. Math can take seconds to read, but hours to understand. In this sense, mathematical proofs are like amazing poetry. The more time you take to decipher the poem, the more layers and more meaning you will be able to take away from it.

So, though this proof seems opaque and impossible the first time you see it, try spending time with it, like some pet you've adopted. You and the pet will take some time before you both get acclimated to each other; but you may find love and companionship at the end of the process.