Lecture 32

Practice with epsilon-delta

32.1 Some notation

Here is some fancy math notation.

1. $\implies$ – an arrow with two tails – means "implies." So for example, $x = 4 \implies x$ is even is a correct use of the symbol $\implies$. The sentence "I am old $\implies$ I am 65" is an incorrect use (because not every old person is 65). If the arrow pointed in the other direction, it would be a correct use.

2. $\forall$ – an upside down A – means "for all," or "for every."

3. $\exists$ – a backward E – means "there exists," or "there is," or "you can find."

Putting this all together, we can re-write the definition of $\lim_{x \to a} f(x) = L$ as follows:

$$\text{If } \forall \epsilon > 0, \exists \delta > 0 \text{ such that } (x \neq a) \& (|x - a| < \delta) \implies |f(x) - L| < \epsilon.$$

The & is just the usual "and" symbol. The main purpose of this notation is to save space and make things shorter-looking; your job is to be able to logically write out what the above condition means.

32.2 Practice problems

Let's practice some more epsilon-delta problems. In the next section, you will see some sample problems worked out. These are the kinds of problems you should be able to do for next time.

Exercise 32.2.1. Let $f(x) = 3x + 7$. Show that whenever $|x - 1| < \frac{2}{3}$, we can conclude that $|f(x) - 10| < 2$.

(Your answer won't be a number. Instead, your answer will be a string of equalities and inequalities that ultimately show that $|f(x) - 10| < 2$. Put another way, you are being graded for your work!)

Exercise 32.2.2. Let $f(x) = 5x + 7$. Show that if $|x - 2| < \frac{1}{5}$, then $|f(x) - 17| < 1$.

Exercise 32.2.3. Let $f(x) = 5x + 7$. Show that if $|x - 2| < \frac{1}{5}$, then $|f(x) - 17| < 3$. (Yes, every number is the same as the previous problem except for the 3.)

Exercise 32.2.4. Let $f(x) = 3x + 7$. Find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 10| < \frac{1}{4}$.

Exercise 32.2.5. Let $f(x) = 3x+7$. Suppose somebody gives you a positive number ϵ. Find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 10| < \epsilon$. (Your δ can be expressed in terms of ϵ.)

Exercise 32.2.6. Let $f(x) = x^2 + 3x + 1$. Suppose $|x - 1| < \frac{1}{12}$. Show that $|f(x) - 5| < \frac{1}{2}$.

Exercise 32.2.7. Let $f(x) = x^2 + 3x + 1$. Can you find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 5| < \frac{1}{4}$?

Exercise 32.2.8. Let $f(x) = x^2 + 3x + 1$. Hiro gives you a number $\epsilon > 0$. Can you find me a number δ so that, whenever $|x-1| < \delta$, we can conclude that $|f(x)-5| < \epsilon$? (Your δ can be expressed in terms of ϵ.)

Exercise 32.2.9. Let $f(x) = 2x^3 + 4x^2 + 7$. Show that if $\delta < \sqrt{\frac{\epsilon}{6}}$, then $|x| < \delta$ implies that $|f(x) - 7| < \epsilon$.

32.3 Sample problems

Exercise 32.3.1. Let $f(x) = 5x + 7$. Show that whenever $|x - 1| < \frac{1}{15}$, we can conclude that $|f(x) - 12| < \frac{1}{3}$.

(Your answer won't be a number. Instead, your answer will be a string of equalities and inequalities that ultimately show that $|f(x) - 10| < 2$. Put another way, you are being graded for your work!)

Solution. Let's first simply $f(x) - 10$ as much as we can.

$$\begin{aligned} |f(x) - 10| &= |5x + 7 - 12| && (32.3.1)\\ &= |5x - 5| && (32.3.2)\\ &= 5|x - 1|. && (32.3.3) \end{aligned}$$

This number could be huge if x is huge; but we are only asked about "whenever $|x - 1| < \frac{1}{15}$, so let's see what we can conclude when this inequality holds. Well, because $|x - 1| < \frac{1}{15}$, we can multiply this inequality by 5 on both sides to get

$$\begin{aligned} 5|x - 1| &< 5 \cdot \frac{1}{15} && (32.3.4)\\ &= \frac{1}{3}. && (32.3.5) \end{aligned}$$

So tracing through the equalities and inequalities we just worked through, we can conclude that

$$|f(x) - 10| < \frac{1}{3}.$$

Which is what the problem wanted us to show!

In a test, just writing out the lines from (32.3.1) to (32.3.5) would get you full credit. To be safe, you may wan to indicate/write that you used the condition that $|x - 1| < \frac{1}{15}$ in line (32.3.4). □

Exercise 32.3.2. Let $f(x) = 5x + 7$. Show that if $|x - 2| < \frac{1}{4}$, then $|f(x) - 17| < \frac{5}{4}$.

Solution.

$$\begin{aligned} |f(x) - 17| &= |5x + 7 - 17| && (32.3.6)\\ &= |5x - 10| && (32.3.7)\\ &= 5|x - 2| && (32.3.8)\\ &< 5 \cdot \frac{1}{4} && (32.3.9)\\ &= \frac{5}{4}. && (32.3.10) \end{aligned}$$

We used that $|x - 2| < \frac{1}{4}$ in Line (32.3.9). □

Exercise 32.3.3. Let $f(x) = 3x + 3$. Find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 6| < \frac{1}{4}$.

Solution.

$$|f(x)-6| = |3x+3-6| \quad (32.3.11)$$
$$= |3x-3| \quad (32.3.12)$$
$$= 3|x-1|. \quad (32.3.13)$$

So we want $3|x-1|$ to be less than $\frac{1}{4}$. Well, if we want the inequality

$$3|x-1| < \frac{1}{4}$$

to be true, it's equivalent to wanting the inequality

$$|x-1| < \frac{1}{12}$$

to be true. So, so long as δ is any number equal to or less than $\frac{1}{12}$, the assumption that $|x-1| < \delta$ means

$$|x-1| < \delta \quad (32.3.14)$$
$$\implies 3|x-1| < 3\delta \quad (32.3.15)$$
$$\leq 3 \cdot \frac{1}{12} \quad (32.3.16)$$
$$= \frac{1}{4}. \quad (32.3.17)$$

So I can give you any δ that's less than or equal to $\frac{1}{12}$. For example, $\delta = \frac{1}{12}$, or $\delta = \frac{1}{13}$. For such a δ, the above work shows that whenever $|x-1| < \delta$, we can conclude that $|f(x)-6| < \frac{1}{4}$. □

Exercise 32.3.4. Let $f(x) = 3x+6$. Suppose somebody gives you a positive number ϵ. Find me a number δ so that, whenever $|x-2| < \delta$, we can conclude that $|f(x)-12| < \epsilon$. (Your δ can be expressed in terms of ϵ.)

Solution.

$$|f(x)-12| = |3x+6-12| \quad (32.3.18)$$
$$= 3|x-6|. \quad (32.3.19)$$

So

$$|f(x)-12| < \epsilon \quad (32.3.20)$$
$$\iff 3|x-6| < \epsilon \quad (32.3.21)$$
$$\iff |x-6| < \frac{\epsilon}{3}. \quad (32.3.22)$$

In other words, so long as $|x-6| < \frac{\epsilon}{3}$, we are guaranteed that $|f(x)-12| < \epsilon$. So we can choose δ to equal $\frac{\epsilon}{3}$, or any positive number less than $\frac{\epsilon}{3}$. □

Exercise 32.3.5. Let $f(x) = x^2 + 5x + 1$. Suppose $|x-1| < \frac{1}{12}$. Show that $|f(x)-7| < \frac{1}{2}$.

Solution.

$$|f(x)-7| = |x^2+5x+1-7| \quad (32.3.23)$$
$$= |x^2+5x-6|. \quad (32.3.24)$$

At this stage, we need to remember a fact I mentioned in class: No matter what, so long as the limit is correct, this polynomial can be factored by $(x-a)$. In our case, $a=1$ (because we are bounding $|x-1|$ in the problem) and sure enough, we can factor so that

$$|x^2+5x-6| = |(x-1)(x-5)|. \quad (32.3.25)$$

Finally, we want things that look like $|x-1|$ to pop up as much as possible in our expressions. This is because $|x-1| < \frac{1}{12}$ is the only fact we are allowed to use about the number x. So for example, $x-5$ can be re-written to be $(x-1)-4$. So let's do that.

$$|(x-1)(x-5)| = |(x-1)((x-1)-4)|. \quad (32.3.26)$$

Next, remember that $|AB| = |A| \cdot |B|$. So

$$|(x-1)((x-1)-4)| = |x-1| \cdot |(x-1)-4|. \quad (32.3.27)$$

Finally, we use the triangle inequality, which tells us that $|C+D| \leq |C| + |D|$. So

$$|(x-1)-4| \leq |x-1| + |4|. \quad (32.3.28)$$

Multiplying both dies of this inequality by $|x-1|$, we see that

$$|x-1| \cdot |(x-1)-4| \leq |x-1| \cdot (|x-1|+4). \quad (32.3.29)$$

We have come a long way to find that (by tracing through all the equations above, along with the one inequality):

$$|f(x)-7| \leq |x-1| \cdot (|x-1|+4). \quad (32.3.30)$$

Because the doodle on the right, $(|x-1|+4)$, is always bigger than or equal to $|f(x)-7|$, if we can guarantee that this doodle is less than ϵ, then we know that $|f(x)-7|$ is also less than ϵ.

Well, we are told that $|x-1|$ is less than $\frac{1}{12}$. So let's see what happens to the doodle:

$$|x-1| \cdot (|x-1|+4) < \frac{1}{12} \cdot \left(\frac{1}{12}+4\right). \tag{32.3.31}$$

Whatever is in the parentheses is certainly smaller than $1+5$, so we can write that

$$\frac{1}{12} \cdot \left(\frac{1}{12}+4\right) < \frac{1}{12} \cdot 5 \tag{32.3.32}$$

$$= \frac{5}{12}. \tag{32.3.33}$$

On the other hand,

$$\frac{5}{12} < \frac{6}{12} \tag{32.3.34}$$

$$= \frac{1}{12}. \tag{32.3.35}$$

Combining all the above, line by line, we conclude that $|f(x)-7| < \frac{1}{12}$, as desired.

(Your solution is the entirety of the work above!) □

Exercise 32.3.6. Let $f(x) = x^2+4x+1$. Hiro gives you a number $\epsilon > 0$. Can you find me a number δ so that, whenever $|x-(-1)| < \delta$, we can conclude that $|f(x)-(-2)| < \epsilon$? (Your δ can be expressed in terms of ϵ.)

Solution.

$$|f(x)-(-2)| = |x^2+4x+1+2| \tag{32.3.36}$$

$$= |x^2+4x+3| \tag{32.3.37}$$

$$= |(x+1)(x+3)| \tag{32.3.38}$$

$$= |x+1||(x+1)+2| \tag{32.3.39}$$

$$\leq |x+1|\left(|x+1|+2\right). \tag{32.3.40}$$

Let's suppose that $|x+1|$ is less than some number C, so $|x+1| < C$. Then we can conclude that

$$|x+1|\left(|x+1|+2\right). < |x+1|\left(C+2\right). \tag{32.3.41}$$

If further $|x+1|$ is less than $\frac{\epsilon}{C+2}$, we conclude that

$$|x+1|\left(C+2\right) < \frac{\epsilon}{C+2} \cdot (C+2) \tag{32.3.42}$$

$$= \epsilon. \tag{32.3.43}$$

So, for any positive number C, choose δ to be any positive number less than C and less than $\frac{\epsilon}{C+2}$. Then the work above guarantees that $|x+1| < \delta$ guarantees that $|f(x) - (-2)| < \epsilon$.

So for example, we could choose δ to be any number less than 1 and less than $\frac{\epsilon}{3}$. □

Exercise 32.3.7. Let $f(x) = 2x^3 + 4x^2 + 3$. Show that if $\delta < \sqrt{\frac{\epsilon}{6}}$ and if $\delta < 1$, then $|x| < \delta$ implies that $|f(x) - 3| < \epsilon$.

Solution.

$$
\begin{aligned}
|f(x) - 3| &= |2x^3 + 4x^2 + 3 - 3| && (32.3.44)\\
&= |2x^3 + 4x^2| && (32.3.45)\\
&= 2|x^3 + 2x^2| && (32.3.46)\\
&= 2|x^2||x+2|. && (32.3.47)
\end{aligned}
$$

We are asked to show that the condition $|x| < \delta$ implies something. Well, if $|x| < \delta$, then—given what we know about δ—we conclude that $|x| < \sqrt{\frac{\epsilon}{6}}$ and $|x| < 1$. So

$$
\begin{aligned}
2|x^2||x+2| &< 2\left(\sqrt{\frac{\epsilon}{6}}\right)^2 (1+2) && (32.3.48)\\
&= 2\left(\frac{\epsilon}{6}\right)(3) && (32.3.49)\\
&= \epsilon. && (32.3.50)
\end{aligned}
$$

The string of equalities and inequalities above shows that $|f(x) - 3| < \epsilon$, as desired. □