Lecture 30

Limit laws, polynomials and rational functions.

30.1 Limit laws: The straightforward ones

Today I'm going to tell you that you can rely on certain laws for computing limits.

Remark 30.1.1. These laws are dissatisfying, because you should demand more: *Why* are these laws valid? We will why later, when we apply the ϵ-δ definition to *prove* these laws.

Limits of constants. If $f(x)$ is a constant function[1] with value C, then

$$\lim_{x \to a} f(x) = C$$

regardless of a.

Limits of x. For the function $f(x) = x$, we have that

$$\lim_{x \to a} f(x) = a.$$

(I encourage you to graph the function $f(x) = x$; then this law will seem "obvious" to you.)

Remark 30.1.2. The first two laws are hopefully not too bewildering; the notation is confusing, but these are meant to be among the simplest examples. I state these just to get our feet wet; it's the next few laws that will really get us going.

[1] This means $f(x) = C$ for some number C. Put another way, the graph of $f(x)$ is just a flat, horizontal line.

Limits scale. *If a limit already exists*, then the limit of the scaled function is the scaled limit of the function. More precisely: *If* $\lim_{x\to a} f(x)$ *already exists*, then for any number m, we have the following:

$$\lim_{x\to a} (m \cdot f(x)) = m \cdot \left(\lim_{x\to a} f(x)\right)$$

Limits add. *If the limits already exist,* then the limit of the sum exists; moreover, the sum of the limits is the limit of the sum.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x) + g(x))$ exists, and

$$\lim_{x\to a} (f(x) + g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$$

Limits multiply. *If limits already exist,* then the limit of the product exists; moreover, the product of the limits is the limit of the product.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x) \cdot g(x))$ exists, and

$$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

Limits divide. *If limits already exist,* then the limit of the quotient exists; moreover, the quotient of the limits is the limit of the quotient (provided the denominator is not zero).

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x)/g(x))$ exists, and

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

so long as $\lim_{x\to a} g(x) \neq 0$.

Remark 30.1.3. The above limit laws have three parts: (i) The *criterion* that certain limits already exist, (ii) The *guarantee* that another limit exists, and (iii) The *formula* of how to compute that other limit.

I wrote all the formulas in such a way that the righthand side of the formula consists of the limits given to exist (by the criterion); the lefthand side is the limit that we are then guaranteed to exist.

Remark 30.1.4. It's important to note that, for every law, the limits are taken at the same point. That is, every limit in sight is taken as x approaches a single number a. So for example, even if I know that $\lim_{x\to a} f(x)$ exists, and that $\lim_{x\to b} g(x)$ exists, I don't know anything about the limits of $f(x) + g(x)$ unless $a = b$. (In which case, I know that a limit exists as $x \to a$.)

30.1.1 Practice with the straightforward limit laws

All of the exercises below could have been solved by "looking at the graphs." But I want you to instead solve them by using the limit laws.

Exercise 30.1.5. Using some of the facts above, convince yourself that if $g(x) = mx$, then[2]

$$\lim_{x \to a} g(x) = g(a).$$

(Hint: Use the function $f(x) = x$ and the scaling law.)

Exercise 30.1.6. Using the limit laws, convince yourself that if $h(x) = x^2$, then

$$\lim_{x \to a} h(x) = h(a).$$

(Hint: Use the functions $f(x) = x$ and $g(x) = x$, along with the product law.)

Exercise 30.1.7. Using some of the facts above, convince yourself that if $h(x) = x^2 + 3$, then

$$\lim_{x \to a} h(x) = h(a).$$

Exercise 30.1.8. Using some of the facts above, show that **limits subtract**.

More precisely, if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then so does $\lim_{x \to a} (f(x) - g(x))$. Moreover,

$$\lim_{x \to a} (f(x) - g(x)) = \left(\lim_{x \to a} f(x)\right) - \left(\lim_{x \to a} g(x)\right)$$

(Hint: Use the fact that limits scale, taking your scaling constant to be $m = -1$, and use the fact that limits add.)

Exercise 30.1.9. Use the limit laws to compute

$$\lim_{x \to 1} \left(\frac{x^2 + 3}{x}\right).$$

What goes wrong when you try to compute the limit as $x \to 0$?

[2] The graph of $g(x)$ is a line of slope m with zero as y-intercept. So even before you knew these limits laws, you should have been able to tell me what the limit as $x \to a$ is!

30.2 Application: Polynomial functions are continuous

Recall the following definition from last time:

Definition 30.2.1. A function f is called *continuous at a* if

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

We talked last time about how continuous functions are "nice." If a function is continuous at a, it means that you can draw the graph of f near a without ever lifting your pencil from the sheet of paper (because the graph won't have a "jump"). If f is continuous everywhere (i.e., at every a) then you can draw the entire graph of f without ever lifting your pencil.

What we'll see now is that *all polynomial functions are continuous.*

Remember that a polynomial function is a function that is made out of adding and multiplying numbers and variables.

Example 30.2.2. For example, $5x^2 + \pi x - \sqrt{2}$ is made by:

- Multiplying $5 \times x \times x$
- Multiplying $\pi \times x$
- Taking the number $-\sqrt{2}$

and adding all three terms together. Another example is $9x^7 - 6x^4 + x$.

I'll use the limit laws to see that $f(x) = 5x^2 + \pi x - \sqrt{2}$ is continuous. First, let's note that for any number a, $f(a)$ is defined—you can just compute it.

Then, note the following:

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(5x^2 + \pi x - \sqrt{2}) && \text{Definition of } f && (30.2.1)\\
&= (\lim_{x\to a} 5x^2) + (\lim_{x\to a} \pi x) - (\lim_{x\to a} \sqrt{2}) && \text{Addition law for limits} && (30.2.2)\\
&= 5(\lim_{x\to a} x^2) + \pi(\lim_{x\to a} x) - \lim_{x\to a} \sqrt{2} && \text{Scaling law for limits} && (30.2.3)\\
&= 5(\lim_{x\to a} x) \cdot (\lim_{x\to a} x) + \pi \lim_{x\to a} x - \lim_{x\to a} \sqrt{2} && \text{Product law for limits} && (30.2.4)\\
&= 5(a) \cdot (a) + \pi(a) - \lim_{x\to a} \sqrt{2} && \text{Limits of } x && (30.2.5)\\
&= 5(a) \cdot (a) + \pi(a) - \sqrt{2} && \text{Limits of constants} && (30.2.6)\\
&= 5a^2 + \pi a - \sqrt{2} && \text{Algebra} && (30.2.7)\\
&= f(a) && \text{Definition of } f && (30.2.8)
\end{aligned}$$

(If it helps, you can plug in $a = 7$, or some other concrete number; the point is that every equality above is true—for the reasons indicated—regardless of what number you choose for a.) So, by tracing through the equalities, we finally see that

$$\lim_{x\to a} f(x) = f(a).$$

This shows that f is continuous.

You can see that you can write a similar list of equalities for *any* polynomial, not just $5x^2 + \pi x - \sqrt{2}$. (Try choosing a polynomial and see for yourself!) So the limit laws, after we use a bunch of them, tell us that polynomial functions are continuous. In other words, this is telling you something you probably already knew: If $f(x)$ is a polynomial function, you can draw its graph without ever taking your pen off the paper.

30.3 Application: Rational functions are continuous where they are defined

Definition 30.3.1. A *rational function* is a fraction of polynomials where the denominator is not zero.

Example 30.3.2. The function $f(x) = \frac{3x^2+9x+1}{x^7-8}$ is a rational function—both the numerator and denominator are polynomials.

So is the function $f(x) = \frac{1}{x}$, because 1 is a polynomial, and so ix x.

Warning 30.3.3. If f is a rational functions, there may be some numbers a for which $f(a)$ is not defined. For example, consider the rational function $f(x) = \frac{9x^2-3}{x+3}$. This function isn't defined at $x = -3$, because then the denominator would equal 0.

Remark 30.3.4. In fact, the only points at which rational functions aren't defined are points at which the denominator equals zero.

Fact. A rational function is continuous everywhere it is defined.

Example 30.3.5. Let $f(x) = 1/x$. Then f is not continuous at zero (it is not even defined at $x = 0$). However, f is continuous everywhere else.

Let's check that f is continuous at 2. To see this, we just need to check all three conditions of the definition.

1. Clearly, $f(2)$ is defined. Then,
2. the fact that limits divide (one of the limit laws!) tells us that $\lim_{x\to 2} f(x)$ exists, and
3. this limit is computed as

$$\lim_{x\to 2} f(x) = \frac{\lim_{x\to 2} 1}{\lim_{x\to 2} x} = \frac{1}{2}.$$

On the other hand $f(2) = 1/2$ by definition of $f(x)$, so we see that

$$\lim_{x\to 2} f(x) = f(2).$$

So we have checked that all three conditions of continuity are satisfied. This means $f(x) = 1/x$ is continuous at $x = 2$.

In fact, let $f(x) = \frac{p(x)}{q(x)}$ be any rational function (so that p and q are polynomials). Let's also suppose that a is a number for which $q(a) \neq 0$. Then

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} \frac{p(x)}{q(x)} && \text{definition of } f && (30.3.1)\\
&= \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} && \text{Quotient law for limits}^3 && (30.3.2)\\
&= \frac{p(a)}{q(a)} && \text{Polynomials are continuous}^4 && (30.3.3)\\
&= f(a) && \text{Definition of } f. && (30.3.4)\\
& && && (30.3.5)
\end{aligned}$$

So, if f is a rational function and is defined at a, then $f(a) = \lim_{x\to a} f(x)$. This proves that any rational function is continuous where it is defined.

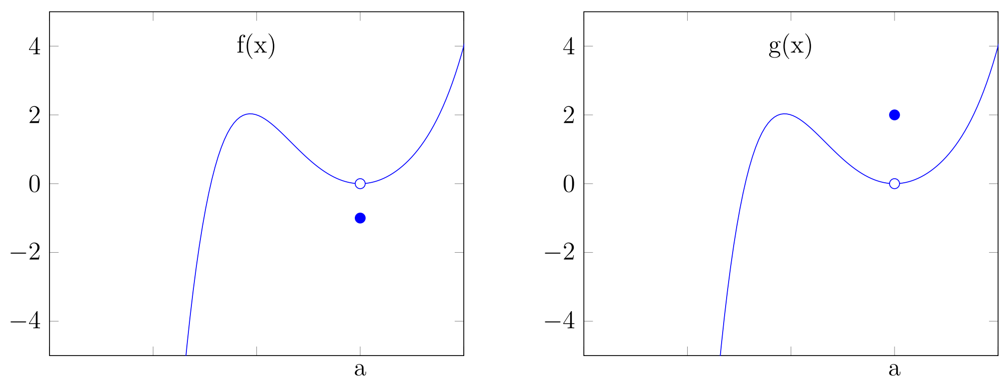
30.4 Puncture law

Puncture law. Let $f(x)$ and $g(x)$ be two functions. Suppose that the two function are equal at all points nearby a (but not necessarily at a itself). Then $f(x)$ has a limit at a if $g(x)$ does, and likewise, $g(x)$ has a limit at a if $f(x)$ does. Moreover,

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Warning 30.4.1. Many calculus textbooks do *not* talk about a "puncture law." In my opinion, this is a bit ludicrous, because about half of the algebraic "tricks" we have to compute limits are dependent on it. I must admit that I made up the term "puncture law," so you may find your peers outside of your class being confused if you use this law.

Example 30.4.2 (A graphical example)**.** On the left is a graph of $f(x)$, and on the right is a graph of $g(x)$.



Note that the value of $f(x)$ and $g(x)$ are different at a (the filled dots are at different heights).[5] But $f(x)$ and $g(x)$ are otherwise identical, so they have the same limit at a. This "obvious" fact is called the puncture law.

Example 30.4.3 (Algebraic example)**.** Let

$$f(x) = \frac{x^2}{x} \qquad \text{and} \qquad g(x) = x.$$

[5] Let me remind you—as I mentioned in class—that the white dot means that the function does *not* take the value of the white dot there. The black dot indicates the value of the function. Often, we write a white dot where it looks like a function wants to take a value, but does not.

Note that $f(x)$ is not defined at $x = 0$, but is equal to $g(x)$ for all other values of x. Thus, the puncture law tells us that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x). \tag{30.4.1}$$

Of course, you know what the righthand side is (by plugging in what $g(x)$ is):

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0. \tag{30.4.2}$$

So, putting (30.4.1) and (30.4.2) together, we see that

$$\lim_{x\to 0} f(x) = 0.$$

In other words (by plugging in the definition of $f(x)$) we find:

$$\lim_{x\to 0} \frac{x^2}{x} = 0.$$

Note that this is an example where the quotient law wouldn't help you, because the limit of the denominator equals zero!

Example 30.4.4 (Rational functions)**.** Let's find the limit

$$\lim_{x\to 2} \frac{(x+1)(x-2)}{x-2}.$$

Note that the function we have is *undefined* when $x = 2$ (because we can't divide by $x - 2$ when $x = 2$). But, we know the following:

$$\frac{(x+1)(x-2)}{x-2} = x + 1 \qquad \textit{so long as } x \neq 1.$$

In other words, the two functions

$$\frac{(x+1)(x-2)}{x-2} \qquad \text{and} \qquad x+1$$

are *equal* away from $x = 1$. Thus, the puncture law tells us

$$\lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to 2}(x+1).$$

Now, let's just compute the righthand side:

$$\begin{aligned}\lim_{x\to 2}(x+1) &= \lim_{x\to 2} x + \lim_{x\to 2} 1 \qquad (30.4.3)\\ &= 2+1\\ &= 3.\end{aligned}$$

(We used the addition law in line (30.4.3).) Putting everything together, we conclude:

$$\frac{(x+1)(x-2)}{x-2} = 3.$$

We're done, but let me streamline everything to show you what you might be able to write on a test:

$$\begin{aligned}\lim_{x\to 2}\frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law}\\ &= \lim_{x\to 2} x + \lim_{x\to 2} 1 && \text{by the addition law}\\ &= 2+1\\ &= 3.\end{aligned}$$

Another solution you might write on a test is:

$$\begin{aligned}\lim_{x\to 2}\frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law}\\ &= 2+1 && \text{because polynomial functions are continuous}\\ &= 3.\end{aligned}$$

Example 30.4.5 (Another rational function)**.** Let's do another rational function example. Let's compute [6]

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9}.$$

This looks very complicated; to use the puncture law, we'd like to find some other function that is equal to $\frac{x^2-2x-3}{x^2-9}$ away from 3. The trick I want you to learn here is that you can *cancel* $(x-3)$ in the top and bottom. This may seem very confusing, because $(x-3)$ doesn't appear anywhere in the function as it's presented. But you'll see that it *does* appear if you factor.

[6] Note that the quotient law doesn't help here, because the limit of the denominator equals zero.

Pro tip. Why do you want to try to cancel $x-3$? It's because we should feel that a term of the form "$x-3$" is what's causing the denominator to equal zero at $x=3$. So it's natural to try and see if, indeed, a factor of $(x-3)$ can pop up in the denominator. More generally, for rational functions, *if you are computing a limit as x approaches a, it is natural to try to find $(x-a)$ as a factor of the top and bottom.*

Warning. If you don't know how to divide or factor polynomials, you should make sure to Google and practice—sometimes we'll need to know how to divide polynomials using long division, or how to factor polynomials through other tricks.

In fact, we can factor both the top and the bottom:

$$\frac{x^2-2x-3}{x^2-9} = \frac{(x-3)(x+1)}{(x-3)(x+3)}.$$

And we see that we can cancel the $(x-3)$ terms! So, when x does not equal 3, our function $\frac{x^2-2x-3}{x^2-9}$ is equal to

$$\frac{x+1}{x+3}. \tag{30.4.4}$$

By the puncture law, we thus conclude the following:

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9} = \lim_{x\to 3}\frac{x+1}{x+3}.$$

And, as we saw, *any rational function is continuous where it is defined.* The rational function in (30.4.4) is defined at $x=3$, so—by the definition of continuity—we can compute the limit simply by plugging 3 into x:

$$\lim_{x\to 3}\frac{x+1}{x+3} = \frac{3+1}{3+3} = \frac{4}{6} = \frac{2}{3}.$$

Putting everything together, we conclude

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9} = 2/3.$$

30.5 Composition law

There is another powerful way to make new functions out of old: Composition. Limits respect composition, too, so long as the outermost function is continuous at the limit of the innermost function:

Composition law. Let $g(x)$ and $f(x)$ be functions, and suppose you know that $f(x)$ is continuous at $\lim_{x\to a} g(x)$. Then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Informally, this means you can "move the limit inside" of f so long as f is continuous where it counts.

Exercise 30.5.1. Using the composition law, and your knowledge that $f(x) = x^2$ is continuous at every point[7], compute

$$\lim_{x\to 3} f(g(x))$$

if g is a function for which $\lim_{x\to 3} g(x) = \pi$.

Warning 30.5.2. To use the composition law, the "outermost" function needs to be continuous where it counts. (Re-read the composition law if this wasn't clear when you first read it!)

30.6 For next time

For next time, I expect you to be able to use the puncture law to compute limits of rational functions. For example, you should be able to compute the following limits:

1. $\lim_{x\to 0} \frac{x^3+3x^2}{x^2}$.
2. $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$.
3. $\lim_{x\to -2} \frac{x^2-4}{x^2+x-2}$.

Also in preparation for next lecture, you should be able to answer the following questions:

(a) What are the *three conditions* you need to check to see whether a function $f(x)$ is continuous at a?

(b) Why is $f(x) = 3/x$ not continuous at zero?

(c) Why is $f(x) = 8/x$ continuous at 5?

[7]You proved this in Exercise 30.1.6!