Lecture 29

Limits, one-sided limits, and continuity; informally

Today, we're going to start talking about the idea of *limits*. I have mentioned at one point that calculus is, roughly, about three ideas: Derivatives, integrals, and limits. We are finally at the last idea.

Remark 29.0.1 (Some motivation.)**.** We have defined the derivative to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

As you know, the righthand side is the value that the difference quotient *approaches* as h goes to zero. So, limits have already made an appearance in our lives, and we already have an intuition of what limits are supposed to be. We'll make these intuitions more precise starting this lecture.

Warning 29.0.2. In real life, the word "limit" can have many meanings. For example, a speed limit tells you that you shouldn't go above a certain speed. Sometimes, a limit is also an "extreme" that you can't get past. Neither of these meanings is what limit means in calculus class. In calculus class, a limit is something we *approach*.

29.1 The basic idea

Let's say that we have a function called $f(x)$. Then we can ask whether $f(x)$ approaches a certain (output) value as x approaches a certain (input) value.

Notation 29.1.1 (Informal.)**.** Let f be a function. The notation

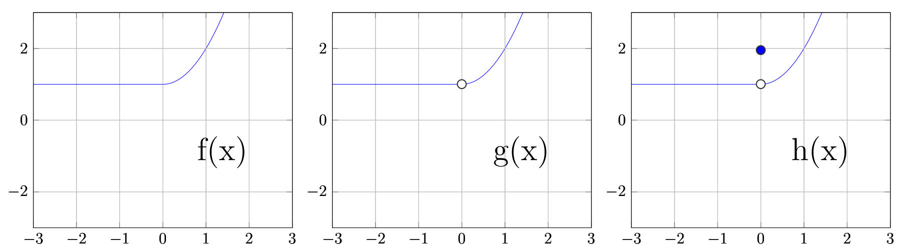
$$\lim_{x \to a} f(x)$$

stands for the number that $f(x)$ approaches as x approaches a.

Warning 29.1.2. Limits might not exist at particular values of a—for example, maybe f doesn't want to approach a single value as x approaches a. We will see examples of this later today.

29.2 Open and solid dots

Let's first review what we mean when we draw certain things. Below are graphs of three functions, f, g and h.



The difference between the three graphs, of course, are the presence of certain dots. Remember that when we draw pictures in math, an "open dot" (e.g., the white dots in the middle picture and the righthand picture) signifies a missing point. In other words, the open dot indicates a point that is *not* on the graph. So the point $(0, 1)$ is *not* a point on the graph of g. In other words, $g(0)$ does not equal 1. In fact, g does not take on any value at $x = 0$. So we say that g is *not defined* at $x = 0$.

Consider, on the other hand, the graph of $h(x)$. The open dot tells us that $h(0)$ does not equal 1. But there is a solid dot at $(0, 2)$. This tells us that $h(0) = 2$. Solid dots indicate that the dot *is* part of the graph.

In contrast, $f(0) = 1$.

Remark 29.2.1. Oftentimes, we draw open and solid dots when the function does something visually unexpected or funny; for example, in the graph of $h(x)$ and $g(x)$, it looks like both functions want to take on the value 1 at $x = 0$. The dots indicate that this is not so. You can see why dots might come up in typical limit problems in calculus class.

29.3 Some simple limits

So let's practice.

Exercise 29.3.1. Using the graph of f above, tell me the following numbers:

(a) $\lim_{x\to 0} f(x)$.

(b) $\lim_{x\to 1} f(x)$.

(c) $\lim_{x\to -2} f(x)$.

Now tell me the above limits by replacing f with g. And then do it for h.

The solutions to the exercise can be "read off" from the graph. As x approaches 0, we see that the blue graph approaches the height of 1. So the first limit is 1. For the next part: As x approaches 1, we see the blue curve wants to (and does) attain a height of 2. Finally, as x approaches -2, we see that the blue graph wants to maintain (and does) attain a height of 1. So

(a) $\lim_{x\to 0} f(x) = 1$.

(b) $\lim_{x\to 1} f(x) = 2$.

(c) $\lim_{x\to -2} f(x) = 1$.

So what about for g and h? In fact, the answers for g and h are *identical* to the answers for f. In other words, f, g, and h have the exact same limits! (Even though, clearly, they are different functions—at $x = 0$, g is not defined, while h is defined there and takes a value of 2 there.)

So we learn a lesson:

Warning 29.3.2. A function does not need to be defined at $x = a$ to have a limit $\lim_{x\to a} f(x)$. (As we saw for g at $x = 0$.)

The value of the function $f(a)$ does not need to equal the limit $\lim_{x\to a} f(x)$. (As we saw for h at $x = 0$.)

29.4 One-sided limits

Sometimes, a function approaches a value from the right; sometimes, the function approaches a value from the left. These values might be different!

Notation 29.4.1 (One-sided limits, informally)**.** If $f(x)$ wants to converge to a value as x approaches a from the right, we call this value the *righthand limit* of $f(x)$ at a, and we denote this value by

$$\lim_{x \to a^+} f(x).$$

(Note the plus sign on the a.)

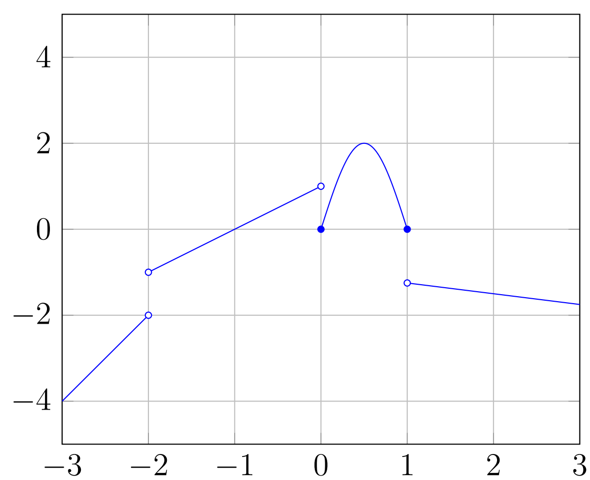
If $f(x)$ wants to converge to a value as x approaches a from the left, we call this value the *lefthand limit* of $f(x)$ at a, and we denote this value by

$$\lim_{x \to a^-} f(x).$$

(Note the *minus* sign on the a.)

A lefthand limit or a righthand limit is called a *one-sided limit*.

Exercise 29.4.2. Below is the graph of a function $f(x)$.



Based on the graph, give your best guest for the following one-sided limits.

(a) $\lim_{x \to -2^-} f(x)$.

(b) $\lim_{x \to -2^+} f(x)$.

(c) $\lim_{x \to 1^+} f(x)$.

(d) $\lim_{x \to 1^-} f(x)$.

(e) What is $f(1)$?

(f) What is $f(0)$?

(g) What is $f(-2)$?

Here are solutions:

(a) $\lim_{x \to -2^-} f(x) = -2$. This one-sided limit asks what value f approaches as x approaches -2 from the left.

(b) $\lim_{x \to -2^+ = -1} f(x)$. This one-sided limit asks what value f approaches as x approaches -2 from the right.

(c) $\lim_{x \to 1^+} f(x) = -1$.

(d) $\lim_{x \to 1^-} f(x) = 0$.

(e) $f(1) = 0$.

(f) $f(0) = 0$.

(g) This is a trick question. f is not defined at -2.

Exercise 29.4.3. Consider the function

$$f(x) = \begin{cases} 0 & x > 0 \text{ and } x \text{ is irrational} \\ 1 & x > 0 \text{ and } x \text{ is rational} \\ 13 & x < 0. \end{cases}$$

Tell me whether $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ exist, and if they exist, what their values are.

29.5 Using one-sided limits

Here is our first theorem about limits. A *theorem* is a true statement that requires an involved proof, and the true statement is so useful that we should[1] know it for future use.

Theorem 29.5.1. The following statements are equivalent:

1. $f(x)$ has a limit at a.

[1] That means you'll be tested on it!

2. Both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, and the one-sided limits agree.

Moreover, in this situation, we can conclude that

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

Remark 29.5.2. The term "equivalent" has a precise meaning here. It means that "if the first statement is true, then the second statement true," *and* that "if the second statement is true, then the first statement is true."

In other words, if f has a limit at a, then it has both one-sided limits there, and they agree. Conversely, if f has both one-sided limits at a and they agree, then f has a limit at a.

Example 29.5.3. Somebody tells you the following information:

$$\lim_{x\to 1^+} f(x) = 3 \qquad \text{and} \qquad \lim_{x\to 1^-} f(x) = 10.$$

Then you know that $\lim_{x\to 1} f(x)$ does not exist, because the two one-sided limits do not agree.

Example 29.5.4. Somebody tells you the following information:

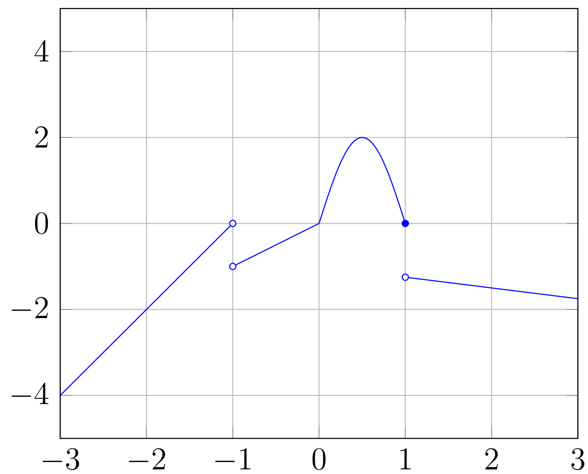
$$\lim_{x\to 2^+} f(x) = 10 \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = 10.$$

Then you know that $f(x)$ *does* have a limit at 2, because the two one-sided limits agree (that is, they have the same value). Moreover, you can conclude that

$$\lim_{x\to 2} f(x) = 10.$$

29.6 Some exercises

Exercise 29.6.1. At which points does the following function *not* have a limit?

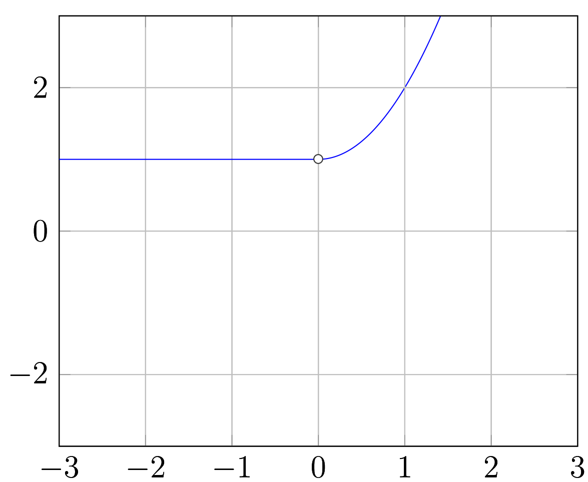


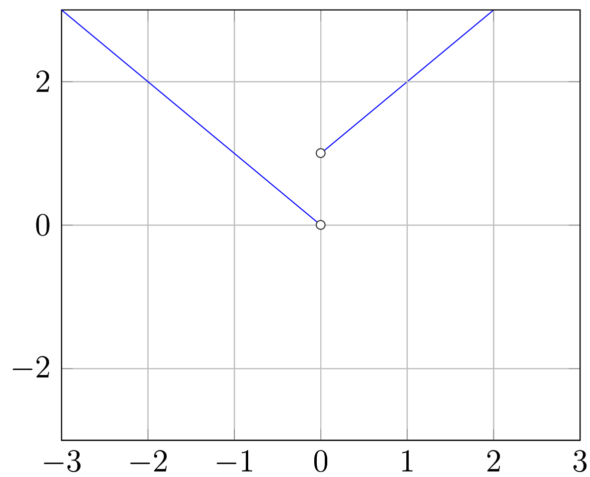
Exercise 29.6.2. For the functions below, determine whether the limit

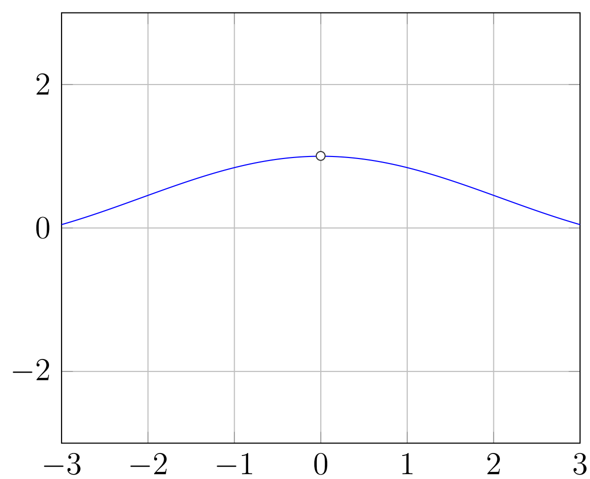
$$\lim_{x\to 0} f(x)$$

exists; and if so, say what the limit is.

(a)



(b) 

(c) 

Exercise 29.6.3. Draw an example of a graph of f satisfying the following properies:

1. $f(1) = \lim_{x\to 1} f(x)$.

2. $f(2) \neq \lim_{x\to 2} f(x)$.

3. f is not defined at $x = 3$ but $\lim_{x\to 3} f(x)$ exists.

4. $\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4^-} f(x)$ exist but $\lim_{x\to 4} f(x)$ does not.

29.7 Limits for functions that aren't presented visually

Below are some functions $q(h)$. Each function q is defined everywhere except at $h = 0$. For each, determine whether the limit

$$\lim_{h \to 0} q(h)$$

exists; and if so, say what the limit is.

1. $q(h) = \begin{cases} h^2 & h \neq 0 \end{cases}$

2. $q(h) = \begin{cases} \sin(h) & h > 0 \\ \cos(h) & h < 0 \end{cases}$

3. $q(h) = \begin{cases} 1 & h \text{ is a rational number and } h \neq 0 \\ 0 & h \text{ is an irrational number} \end{cases}$

Remark 29.7.1. Recall that a rational number is a number that can be expressed as a fraction—things like $-2/7$, or 13, or $5/6$. An irrational number is a real number that is not a fraction. For example, $\sqrt{2}$ or π.)

Remark 29.7.2. A function is called *piecewise defined* when it is defined in the following format:

$$q(h) = \begin{cases} \text{blah blah} & \text{some condition on } h \\ \text{blahbitty blah} & \text{some other condition on } h \\ \text{Rob Loblaw} & \text{perhaps another condition on } h \end{cases}$$

We tend to define functions using the above format when it's not easy to define the function in one fell swoop. For example, the function (3) above means that $q(h)$ equals 1 when h is a non-zero rational number, and equals 0 when h is an irrational number.

29.8 Continuity

Now, you probably get the feeling that the kinds of graphs we've drawn today are different from the kinds of graphs we've drawn earlier in this course. (Earlier in this course, we didn't need open dots or closed dots, for example.) It turns out there's a name for the kinds of phenomena that we are seeing today:

Definition 29.8.1. A function f is called *continuous at a* if

1. $f(a)$ is defined,
2. $\lim_{x\to a} f(x)$ exists, and
3. $\lim_{x\to a} f(x) = f(a)$.

If a function is continuous at a, you can think of f as looking "nice" near a. Here's one intuition: If a function is continuous at a, it means that you can draw the graph of f near a without ever lifting your pencil from the sheet of paper (because the graph won't have a "jump"). If f is continuous everywhere (i.e., at every a) then you can draw the entire graph of f without ever lifting your pencil.

As it turns out, most of the functions that we've studied in this class have been continuous everywhere they've been defined; this is why we can draw their graphs without having to "jump" or lift a pencil.

29.9 Real-world examples of discontinuity

But in real life, there are plenty of functions that aren't continuous. More on that next time, potentially.

29.10 For next time

Make sure you can do the exercises of these lecture notes.