Lecture 24

Applications of integration

When we studied derivatives, we saw that "the slope of the tangent line at a" had to do with "the rate of change at a." In other words, the geometric idea of slope had real-world implications (by allowing us to understand how things are changing at a given moment).

Likewise, today we'll study how the geometric notion of "area" can actually represent meaningful quantities in the real world.

24.1 Units of derivatives

First, let's talk a little about units (like meters, seconds, liters, dollars et cetera).

Let's say that $f(t)$ is a function telling us the *position* of something—say, in meters—away from some reference point. Let's say that the input variable t is time, measured in seconds. Then when we compute the slope of a secant line

$$\frac{f(t+h)-f(t)}{h}$$

the physical quantity in the numerator is measured in meters, while the quantity in the denominator is measured in seconds. In other words, the above slope has natural units, called

$$\frac{\text{meters}}{\text{seconds}}$$

otherwise known as meters per second. And indeed, these are the units with which we measure speed and velocity. (If f were to output a number in terms of miles, and if t were measured in hours, the slope would have units of miles-per-hour.) This is confirmation, using the physical idea of units, that the derivative of a position-in-time function can be interpreted as velocity.

Example 24.1.1. If $v(t)$ is a function whose input variable t is measured in seconds, and whose output value $v(t)$ is the velocity of something in motion, measured in meters-per-second, then the units for the derviative $v'(t)$ would be in

$$\frac{\text{meters-per-second}}{\text{seconds}}.$$

In other words, the units would be "meters per second per second," often abbrevaited as "meters per second squared," or m/s^2. If you have taken a physics class, you may recognize this as a unit for acceleration—it measures how velocity is changing with respect to time.

Example 24.1.2. If $P(d)$ is a function that tells you the water pressure underwater (measured in millibars) at depth d (measured in meters), then the derivative of P with respect to d would have units

$$\frac{\text{millibars}}{\text{meter}}$$

or millibars-per-meter. This tells you the rate of change with pressure with respect to depth.

24.2 Units of integrals

What kinds of units pop out from integrals?

24.2.1 The integral of velocity is distance (or position)

Let's say that $v(t)$ is a function that tells you the velocity of an object (measured in meters-per-second) as a function of the time t (measured in seconds). Remember that the integral is obtained by approximating the area between the graph of $v(t)$ and the t-axis by drawing a bunch of rectangles. So what units does the area of a rectangle have?

If we draw a rectangle of height v (measured in meters-per-second) and width Δt (measured in seconds), the product $v \cdot \Delta t$ has units

$$\frac{\text{meters}}{\text{seconds}} \cdot \text{seconds}.$$

The units of seconds cancel, leaving us with units of meters. In other words, the area under the graph of $v(t)$ naturally has units of meters—i.e., the integral must represent some sort of distance!

This makes sense: Velocity times time equals distance traveled. The rectangles we draw when computing Riemann sums represent our attempts at computing distance traveled by replacing $v(t)$ (where velocity could be changing all the time) with a bunch of functions that look constant (as though velocity were constant over the intervals Δt).

The upshot. If $v(t)$ is a velocity function and t is a time variable, then

$$\int_a^b v(t)\,dt$$

represents the *distance* traveled between time a and time b. Informally, *the integral of velocity (with respect to time) is distance.*

Remark 24.2.1. This makes sense if you think of "integrals" (i.e., antiderivatives) as the reverse operation of "derivatives." You take the derivative of position to obtain velocity; so the antiderivative of velocity is position.

24.2.2 Density and mass

Here is another common situation where integrals come up. Let's suppose that you have a rod, but the rod is made up of a non-uniform material. We'll be concerned with the density of the rod, which is a measure of how much mass is contained in a particular portion of the rod. (For example, the rod may be made of heavier material at one end than at the other end.)

Then we may be given a function $\rho(l)$ which tells us the density of the rod l meters away from one end of the rod.[1] We'll be interested in density as measured in "kilograms per meter."[2] If you think about it for a moment: If you know the density of a rod in terms of kilograms per meter, then if you multiply this density by the length of the rod (meters), you should obtain the mass of the rod itself. This would be very much true if the rod had uniform/constant density, but in our situation, ρ changes with respect to l.

[1] ρ is a Greek letter, read "rho." ρ is the lower-case form. P is the capital form. I don't know why, but it is common to denote density by this Greek letter.

[2] In real life, we most often measure density in kilograms per cubic meter, but we only care about how density changes as we move along the length of our rod, so we'll use kilograms per meter.

But what if we compute the integral

$$\int_a^b \rho(l)\, dl\, ?$$

Again approximating this integral using rectangles (using a Riemann sum), we see that each rectangle's area represents a quantity in units of

$$\frac{\text{kilograms}}{\text{meters}} \cdot \text{meters}.$$

In other words, each rectangle's area has units of kilograms. (Just as with velocity, each rectangle represents an approximation where we pretend that density is constant over the width of the rectangle.) Adding all the areas of these rectangles up, we obtain something in units of kilograms—and it is the mass of the portion of the rod living over the interval $[a, b]$.

Upshot. The integral of density (with respect to length) is mass.

Remark 24.2.2. In fact, this kind of thinking works for all kinds of situations when you replace density with concentration per unit length. (You can think of density as the concentration of "mass" per unit length.) For example, if a straight piece of string was soaked with a chemical, and the concentration of the chemical is changing along its length, the integral of this concentration (with respect to length) would represent the total amount of chemical soaked into the string.

Remark 24.2.3. You might be a little dissatisfied that we've only talked about rods and strings. In real life, we might be interested in concentrations of chemicals measured over swaths of land (i.e., an actual region, now just a string), or in density of substances with actual volume. Indeed, you can take integrals over things like swaths of land and volumes of things, but we won't talk about that in this class. You'll learn how to do such things if you take a "multivariable calculus" class, which is sometimes called "Calculus III." (Also, a dirty secret: I think that Calculus III can be taken successfully without ever taking Calculus II.)

24.2.3 Getting used to wonky units

Sometimes, we have to get used to wonkier units.

For example, let's say $P(t)$ represents the population (of a country, say) at time t. So P is measured in units of persons; and let's say we measure t in years. Then what would

$$\int_a^b P(t)\, dt$$

represent? Whatever it represents, it will have units of “person-years.” (You can think of this as “Person×Years.”) If you aren’t used to this kind of thing, these units can seem quite strange.

But they can be quite useful. For example, let’s say that on average, a person in our country consumes 25MWh (megawatt-hours)[3] of natural gas per year in energy usage. (This is about the average for United States as of 2015.) In other words, our country consumes about 25MWh’s worth of natural gas per person per year.

So if we multiply the above integral by 25, we obtain something in units of

$$\frac{\text{MWh}}{\text{person-year}} \times \text{person-year}.$$

In other words, we’ll determine the total amount of natural gas (in terms of megawatt-hours) that we as a country consumed between time a and time b.

24.2.4 Work examples

In physics, *work* is defined as “force times distance.” For example, let’s say you want to lift a box weighing 10 kilograms 5 meters. It turns out that the force of gravity is always acting on that box, and this force can be computed to be 98 Newtons. If you want to lift the box 5 meters, then the work required to that is

$$98 \text{ Newtons} \times 5 \text{ meters} = 490 \text{ Newton-meters}.$$

In general, if you want to lift an object weighing W kilograms a height of h meters, the work required is

$$\text{force times distance} = W \times 9.8 \text{ Newtons} \times h \text{ meters} = 9.8hW \text{ Newton-meters}.$$

This should gice some intuition about the word work—the more something weighs, of course it takes more “work” to lift it; and the farther up you want to lift it, of course it takes more “work.”

Remark 24.2.4. It turns out that Newton-meters is also the unit for energy; in other words, work is measured in energy units. And this is actually the real use of work in physics.

As an example, let’s suppose that you lift up a 10-kilogram box by 5 meters, so you do 490 Newton-meters of work (as we showed above). Now let’s say you drop the

[3] Megawatt-hours is a common unit of measuring energy. As a point of comparison, in the US, as of 2014, it’s estimated that the average person uses about 13MWh of electricity per year. (This is more than quadruple the global average electricity use.)

box. It turns out that, when the box hits the ground again, it has a kinetic energy of exactly 490 Newton-meters (ignoring air resistance for the discussion). Kinetic energy is defined as one-half of mass times velocity-squared; it is the energy of a moving object.

There are many situations where the force in question is not constant.

Example 24.2.5. For an ideal spring, there is a number k so that when the spring is stretched x meters from its natural state, you need to exert a force of kx Newtons to maintain the spring's stretched state. This is called Hooke's Law; you usually learn about it in a course in physics. The number k is called the spring constant.

How much work does it take to stretch a spring from 0 to 0.2 meters from its natural state?

Well, at any given moment when the spring is stretched x meters, one must exert kx Newtons of force. To stretch this a further Δx meters, the work to do that would be $kx \times \Delta x$.

So, if we were to approximate this process using tinier and tinier intervals of stretching, we would end up with an expression like

$$\sum_{i=1}^{n} kx_i \Delta x$$

where we are using n tiny intervals of stretch, $x_1 = 0 + \Delta x$ and $x_n = 0.2$. Of course, by definition, if we take $n \to \infty$, we find the integral:

$$\int_0^{0.2} kx\, dx.$$

This integral becomes

$$\frac{1}{2}kx^2 \Big|_0^{0.2}.$$

24.2.5 Pandemic examples

The most common graph you've seen on the news during the pandemic has to do with "new infections." If you model this bar graph using a function $f(t)$, then $f(t)$ can be measured in units of "persons per day" (measuring the number of people testing positive per day) and t can be measured in units. Then the integral

$$\int_a^b f(t)\, dt$$

has units of (persons per day)×(days); that is, units of "persons." It measures the number of people who tested positive between time a and time b.[4]

Similarly, if $g(t)$ is a function modeling the number of new hospitalizations per day, the integral $\int_a^b g(t)\,dt$ measures the total number of hospitalizations between time a and time b. Sometimes, we are more interested in $g(t)$ than in the integral, because hospitals may be most afraid of a "rush" of new patients—that is, a hospital may be able to handle 10,000 hospitalizations over the course of a year, but certainly not if they all come at once.

Regardless, let's let $G(t)$ represent the total number of people in the hospital for COVID-19 at time t. Then $\int_a^b G(t)\,dt$ is measured in person-days. This is a potentially useful measure of how much of our hospital resources COVID-19 is occupying. After all, this integral takes into account the fact that a patient who spends 20 days in the hospital may require more hospital resources than a patient who spends only 1 day in the hospital.

24.3 Exercises and examples

Exercise 24.3.1. It is estimated that the average person expels about 500 liters of carbon dioxide per day, or 182,500 liters per year. So, the rate of carbon dioxide produced by respiration is about

$$182{,}500 \text{ liters per person per year.}$$

We can model the world's human population using the function

$$P(t) = 7.3 \times 2^{\frac{t}{50}}$$

where t is measured in years, P is measured in *billions* of people, and we'll take January 1, 2015 as our starting point $t = 0$.

According to this model, how many liters of carbon dioxide will have been emitted by human beings (by breathing alone) between January 1, 2015 and January 1, 2025?

You should express an exact form of your answer (it will involve natural logs) before giving a decimal form.

Exercise 24.3.2. (The following model is entirely fictional, though Mechanical Turk is a real thing. I do not know what the average hourly wage on Mechanical Turk is.)

[4]This "number of people who tested positive" is a bit misleading; it is possible that some people test positive multiple times.

On the website Mechanical Turk, we can model the number of people $P(t)$ working at a particular hour t using the function

$$P(t) = 12000 + 10000\sin(\frac{2\pi}{24}t)$$

where t is measured in hours from midnight.

On average, a worker for Mechanical Turk is paid 1.3 dollars per hour. (So the website pays 1.3 dollars per hour per worker.)

Based on this, how much money did the website Mechanical Turk have to pay its workers between 9 AM and 6 PM?

You should express an exact form of your answer (it will involve sines or cosines) before using a calculator to give a decimal form.

Exercise 24.3.3. Some rideshare apps use "surge" pricing to change the cost of a hailed ride. Let's assume that Company U pays its drivers a wage that depends on this surge pricing, and that we can model worker wages by the function

$$W(t) = 9 + 3\cos(\frac{2\pi}{12}(t-8)).$$

Here, W is measured in dollars per hour and t is measured in hours past midnight. (So for example, at 8 AM, a driver is earning 12 dollars per hour. At 2 PM, a driver is earning 6 dollars per hour.)

Based on this model, if a driver works from 9 AM to 5 PM, how much money do they earn from Company U?

(A similar model can be used to calculate costs of electricity, which typically fluctuate based on time of day, though with electricity, we must also couple the "surge" pricing with the amount of electricity we actually use to determine price.)

24.4 For next time

Practice integrals.

If somebody tells you the units that $f(x)$ and x are measured in, you should be able to tell me what units $\int_a^b f(x)\,dx$ has, and what units $f'(x)$ has.

By the way, because someone asked: An exact answer to the first exercise is given by

$$7.3 \times 182,500 \times 1,000,000,000 \times \tfrac{50}{\ln 2}\left(\sqrt[5]{2}-1\right) \tag{24.4.1}$$

$$= 666,125,000,000,000,000 \tfrac{\sqrt[5]{2}-1}{\ln 2} \tag{24.4.2}$$