Lecture 21

Integration and the Fundamental Theorem of Calculus

Given a function f, along with an interval $[a, b]$, we saw last time that we can *approximate* the area under the graph of f.[1] We did this by choosing n rectangles, so that their widths were given by $(b - a)/n$, and by choosing a height of each rectangle as dictated by f. We ended up with a summation that looked like

$$\sum_{i=0}^{n-1} f(x_i)\frac{b-a}{n} \qquad \text{(Lefthand rule).}$$

Now, these approximations should get better the more rectangles that we use—that is, the bigger n is. In this class, we will define the integral as follows:

Definition 21.0.1 (The integral)**.** The integral of f from a to b is defined to be:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\frac{b-a}{n}.$$

We denote this limit by

$$\int_a^b f(x)dx.$$

In words: This is the limit of the numbers we obtain from the Riemann sum when we let n grow larger and larger.

[1] "Under" is informal; remember, this is the area between the graph of f and the x-axis, computed with "sign," where regions under the x-axis are declared to have *negative* areas.

21.1 Getting used to, and reading, the notation

This is the first time you've seen the notation

$$\int_a^b f(x)dx.$$

Let's dissect this.

First, the above collection of symbols can be described as:

"The integral of $f(x)$ from a to b."

If you were reading the symbols out loud like an automated reader, you would say "the integral from a to b of f of x, dx." [2]

Second: $\int$ is called "the integral symbol." And a and b are called the *bounds* of the integral. The interval $[a, b]$ is sometimes called the *region of integration.*

Third: $f(x)$ is referred to as the *integrand* of the integral.

Finally, let's talk about where this notation comes from. Well, we saw a Riemann sum:

$$\sum_{i=1}^{n} f(x_i)(b-a)/n$$

which we could rewrite (by thinking of $(b-a)/n$ as a change in x) as

$$\sum_{i=1}^{n} f(x_i)\Delta x.$$

Now imagine being supremely lazy and neglecting to write the super and subscripts for Σ:

$$\sum f(x_i)\Delta x.$$

Remember I told you earlier that Σ is a Greek letter that turned into S. Imagine being so lazy that you start writing your S really quickly, until your writing started to look like this:

$$\int f(x_i)\Delta x.$$

At this point you've become so lazy that you don't know that the i mean any more, so let's drop that:

$$\int f(x)\Delta x.$$

[2] Analogously, you would describe $f'(x)$ as the derivative of f at x, but if you read the symbols out loud, you would say "f prime of x".

And Δ is a Greek letter that has now-a-days turned into D. For no good reason, let's make it lower-case:

$$\int f(x)dx.$$

You should remember that you're taking the area between $x = a$ and $x = b$, so let's at least remember that:

$$\int_a^b f(x)dx.$$

And this is one way to reason out how this notation came about. This notation actually has very good reasons to exist, especially because the symbol dx actually has an incredibly fancy meaning in the math community. But we're not ready to confront that meaning in this class. (Nor will we be ready until you have had enough multivariable calculus.)

21.2 Definitions versus intuitions

Remember that I make a hubbub about what is a definition, and what is an intuition. Here is a table of intuitions and definitions for our three most important ideas:

Term	Definition	Intuition
Derivative	The limit of a difference quotient[a]	The **slope** of the line tangent to the graph of f at x.
Integral	The limit of values of Riemann sums[b]	The **area** between the graph of f and the x-axis.

[a]We define $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

[b]We have defined $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\frac{b-a}{n}$.

Why do I make a hubbub? Intuition is how you should think about things: They guide you in solving problems. *Definitions* give content to your intuition, and allow you to actually *prove* things. For example, even if you have an intuition for derivatives, you would be hard-pressed to prove that $(\sin x)' = \cos x$ without the definitions of both limit and of derivative!

I emphasize this. Remember, the equation $(\sin x)' = \cos x$ isn't true just because some teacher told you it was; it's true because you can *prove* it, and the proof doesn't consist of people waving their hands about what the slope of the tangent line should be—the proof consists of using definitions and logical deductions to get from Step A to Step B. This is the heart of mathematics.

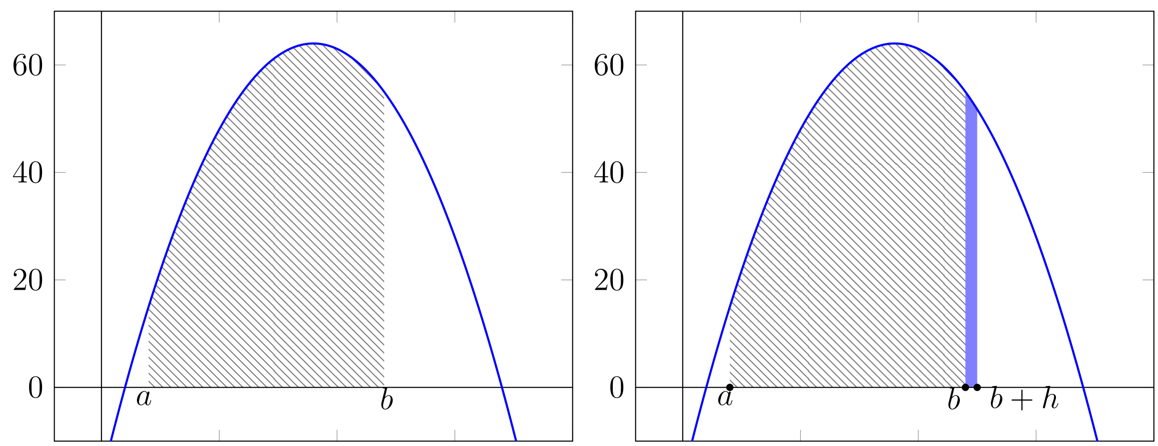
That being said, just as with derivatives, I will only ask you to prove *some* of the rules of integration, and other rules, I will (hypocritically) ask you to just take them on faith. And, as I mentioned before, your *intuition* of thinking about integral as area will help you most.

21.3 The rate of change of area

Let's say you have a function $f(x)$, and you want to compute

$$\int_a^b f(x)dx.$$

As I've advocated, let's think of this as the area. Now let's say you want to nudge the region of integration just a little bit bigger—say by some number h.



In other words, suppose we want to compare

$$\int_a^b f(x)dx \qquad \text{with} \qquad \int_a^{b+h} f(x)dx.$$

The difference is (using our intuition) the area of $f(x)$ between b and $b+h$—this is the darker shaded region in the picture above. In other words, the difference is

$$\int_a^{b+h} f(x)dx - \int_a^b f(x)dx = \int_b^{b+h} f(x)dx = \text{area of darker shaded region.} \quad (21.3.1)$$

Exercise 21.3.1. (a) By approximating the shaded region using a single rectangle of width h and using the lefthand rule for this single rectangle, approximate

$$\int_a^{b+h} f(x)dx - \int_a^b f(x)dx.$$

(b) Using your answer from part (a), write down an approximation for

$$\frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h}.$$

Intuitively, would you expect your approximation to be better or worse as h shrinks?

(c) Based on part (b), make a guess as to what

$$\lim_{h\to 0} \frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h}$$

should be in term of f and b.

(d) Note that the area $\int_a^b f(x)dx$ should change if we change the bounds of integration—and in particular, if we change b. So, keeping a unchanged for now, let's define a function F as follows:

$$F(b) = \int_a^b f(x)dx.$$

What can you say about the derivative of F at b?

21.4 A guide for the previous exercise

How can we approximate the area of the grey region? As the exercise suggests, let's use a single rectangle, and the lefthand rule. Then the grey region can be approximated by a single rectangle of width h and height $f(b)$. So

$$\int_b^{b+h} f(x)dx \approx \text{height} \times \text{width} = f(b) \cdot h. \tag{21.4.1}$$

Here, the symbol $\approx$ means "approximately equals." Putting (21.3.1) and (21.4.1) together, we find:

$$\int_a^{b+h} f(x)dx - \int_a^b f(x)dx \approx f(b) \cdot h.$$

So, dividing both sides by h, we have that

$$\frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h} \approx f(b).$$

Of course, if h is a tiny number, this approximation should get better and better. In other words, that $\approx$ symbol will behave more like an $=$ symbol as h shrinks. Hmm. We've learned what it means to let h approach 0—we should take a limit. Thus, knowing that $\approx$ should become closer and closer to becoming an equals sign as h goes to 0, we seem to get

$$\lim_{h \to 0} \frac{\int_a^{b+h} f(x)dx - \int_a^b f(x)dx}{h} = f(b). \tag{21.4.2}$$

Remark 21.4.1. Let's take a moment to parse this. This is saying the following: We can think of area—i.e., of $\int_a^b f(x)dx$ as something that depends on b.[3] And (21.4.2) is telling us that the *rate of change at b*—that is, the derivative of the area function at b—seems to be very close to being $f(b)$ itself.

Let's really hone in on this observation. Let's say $F(b)$ is the function that tells us the area of f between a and b. Then (21.4.2) seems to be telling us that

$$\frac{dF}{db}(b) = f(b).$$

In other words, whatever function F measures *area*, it seems to be a function whose derivative recovers f.

[3]Intuitively, if we increase b, we are increasing the region of integration, and the area we're interested in clearly changes. So the number, area, changes depending on b.

21.5 The fundamental theorem of Calculus

All the “seems” and “$\approx$” and “approximates” on the previous page was maddeningly imprecise, suggestive, incomplete. Thanks to Isaac Newton and Gottfried Wilhelm Leibniz, it turns out that what “seems” actually “is.”

Theorem 21.5.1 (The fundamental theorem of calculus)**.** Let $F(x)$ be any function such that

$$F'(x) = f(x).$$

Then, so long as f is continuous on the interval $[a, b]$, we have:

$$\int_a^b f(x)dx = F(b) - F(a).$$

A function $F(x)$ such that $F'(x) = f(x)$ is called an *antiderivative* of $f(x)$. What the fundamental theorem says is that we can compute an integral of f (which was defined in a way that had nothing to do with derivatives!) by finding antiderivatives of f.

21.6 Preparation for next time

21.6.1 Finding antiderivatives

I expect you to be able to do the following:

(a) Let $f(x) = 3$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(b) Let $f(x) = 5x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(c) Let $f(x) = x^3$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(d) Let $f(x) = \sin x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(e) Let $f(x) = \cos x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(f) Let $f(x) = e^x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

(g) Let $f(x) = 1/x$. Find a function $F(x)$ so that $F'(x) = f(x)$.

21.6.2 Finding areas

I also expect you to be able to use the fundamental theorem of calculus (and the intuition that the integral is the area) to do the following problems:

(a) Let $f(x) = 3x^2$. Find the area of $f(x)$ along the interval $[1, 4]$.

(b) Let $f(x) = e^x$. Find the area of $f(x)$ along the interval $[0, \ln 3]$.

(c) Let $f(x) = \cos x$. Find the area of $f(x)$ along the interval $[0, \pi/4]$.

21.6.3 Solutions to finding atiderivatives

Here are some solutions.

(a) We have seen functions whose derivatives are just constants—for example, if $F(x) = 7x$, then we know $F'(x) = 7$. So if we want $F'(x)$ to equal 3, we can take $F(x) = 3x$.

However, we could also choose $F(x) = 3x + 13$. For then the derivative is again $F'(x) = (3x') + (13)' = 3 + 0 = 3$. Indeed, we could add any number (constant) to $3x$ to obtain a function that satisfies $F'(x) = f(x)$.

(b) Let $f(x) = 5x$. Find a function $F(x)$ so that $F'(x) = f(x)$. We know that the derivative of x^2 is $2x$, so if we just multiply x^2 by the correct multiple, we can engineer the derivative to become $5x$. So let's try

$$F(x) = \frac{5}{2}x^2.$$

Then by the power rule, we indeed find that $F'(x) = \frac{5}{2} \times 2 \times x = 5x$.

And, just as in the previous problem, if we add a constant, so we try

$$F(x) = \frac{5}{2}x^2 + 999$$

for example, then we still have $F'(x) = 5x^2$.

Do you see a pattern? There are *infinitely many* possible choices for F, always. And they'll always differ by some constant.[4]

(c) We know that x^4 has derivative $4x^3$. So just as in the previous problem, let's try

$$F(x) = \frac{1}{4}x^4.$$

Then $F'(x) = \frac{1}{4} \cdot 4x^3 = x^3$. And, as before, we can add any constant to $\frac{1}{4}x^4$ to obtain another function whose derivative is given by $f(x) = x^3$. For example,

$$F(x) = \frac{1}{4}x^4 + \pi$$

is a valid solution.

[4] In fact, we proved after the mean value theorem that if two functions have the same derivative, the two functions must differ by a constant.

(d) Let $f(x) = \sin x$. As before, we must utilize our knowledge of derivatives to realize that $F(x) = -\cos(x)$ is a valid choice. And, as before, we can also take something like $F(x) = -\cos(x) + 23$ (or any constant) as a viable candidate.

(e) Let $f(x) = \cos x$. Then we can take $F(x) = \sin x$, plus any constant we desire.

(f) Let $f(x) = e^x$. We can take $F(x) = e^x$, plus any constant we desire.

(g) Let $f(x) = 1/x$. We can take $F(x) = \ln x$, plus any constant we desire.

At this point, you may be tired of me telling you how we can add any constant we want. For this reason, we will often write

$$F(x) = \ln x + C$$

instead of writing out "$F(x)$ could be $\ln x$ plus any constant we desire." The capital C stands for "constant."

21.6.4 Solution to finding areas

(a) Let $f(x) = 3x^2$. Find the area of $f(x)$ along the interval $[1, 4]$.

In this problem, $[a, b] = [1, 4]$—that is, $a = 1$ and $b = 4$. The area is computed by finding the integral

$$\int_a^b f(x)dx = \int_1^4 3x^2 dx.$$

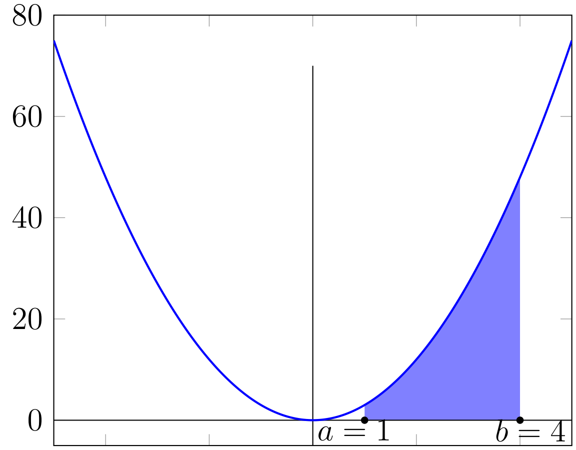
(Note that we never remove the "dx" when we solve problems; we'll see a use for this later on.) By the fundamental theorem of calculus, we know that

$$\int_1^4 3x^2 dx = F(4) - F(1)$$

if F is any function satisfying $F'(x) = 3x^2$. Well, we can find such a function. Let $F(x) = x^3$. So, by the fundamental theorem of calculus (FTC), we have

$$\int_1^4 3x^2 dx = F(4) - F(1) = (4)^3 - (1)^3 = 64 - 1 = 63.$$

How cool is that? You just proved that the region shaded below:



has area given by 63!

As we see from above, the steps to finding the area/integral $\int_a^b f(x)dx$ are: (1) Find an antiderivative for f, then (2) plug in b and a into the antiderivative, and take difference.

(b) Let $f(x) = e^x$. Find the area of $f(x)$ along the interval $[0, \ln 3]$. Again, area is given by

$$\int_0^{\ln 3} f(x)dx.$$

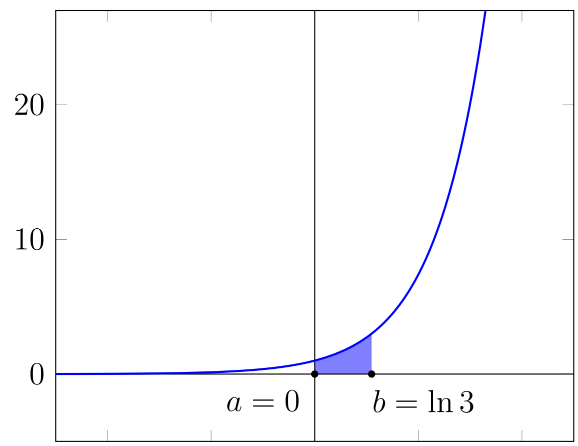
By the fundamental theorem of calculus, if we find some $F(x)$ such that $F'(x) = f(x)$, then

$$\int_0^{\ln 3} f(x)dx = F(\ln 3) - F(0).$$

Well, e^x is its own derivative, so we can take $F(x) = e^x$. Then

$$\int_0^{\ln 3} f(x)dx = F(\ln 3) - F(0) = e^{\ln 3} - e^0 = 3 - 1 = 2.$$

This proves that the region shaded below

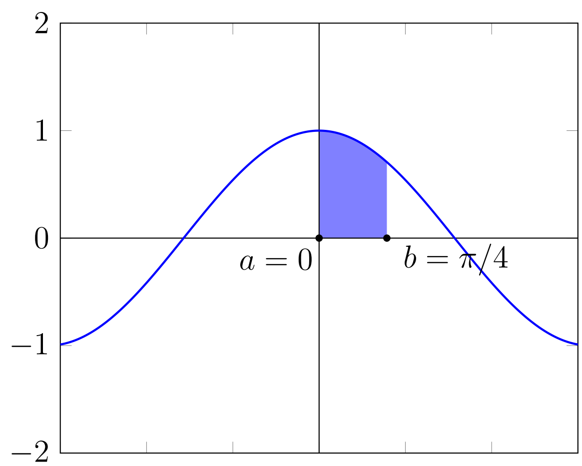


has area given by 2.

(c) Let $f(x) = \cos x$. Find the area of $f(x)$ along the interval $[0, \pi/4]$. Let $F(x) = \sin x$. Then

$$\int_0^{\pi/4} \cos(x)dx = F(\pi/4) - F(0) = \sin(\pi/4) - \sin(0) = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.$$

So you've proven that the area of the region shaded below



is $\frac{\sqrt{2}}{2}$.