Lecture 16

Taylor Polynomials

16.1 Derivatives of polynomials are easy to compute at certain places

Exercise 16.1.1. Consider the function

$$T(x) = 5 + (x - \pi) + 9(x - \pi)^2 + 17(x - \pi)^3 + 19(x - \pi)^4.$$

This is a degree four polynomial.

(a) Compute $T(\pi)$. **Hint:** Do *not* expand out the powers of $x - \pi$.

(b) Compute $T'(\pi)$. **Hint:** Do *not* expand out the powers of $x - \pi$.

(c) Compute $T''(\pi)$. **Hint:** ... Guess the hint.

(d) Compute the *third* derivative of T at $x = \pi$. (The third derivative is the derivative of the second derivative.) This is sometimes written $T'''(3)$, or sometimes $T^{(3)}(\pi)$.

(e) Compute $T^{(4)}(\pi)$. That is, compute the fourth derivative of T at $x = \pi$.

Exercise 16.1.2. Suppose you have a degree four polynomial of the form

$$T(x) = b_0 + b_1(x - a) + \frac{b_2}{2}(x - a)^2 + \frac{b_3}{3!}(x - a)^3 + \frac{b_4}{4!}(x - a)^4$$

where $a, b_0, b_1, b_2, b_3, b_4$ are real numbers.

(Some notes:

- Remember that 4! is a "factorial." It is a shorthand for the expression $4 \times 3 \times 2 \times 1$. Likewise, $3! = 3 \times 2 \times 1$.
- If it helps, you can choose to replace b_0, b_1, and so forth with concrete numbers like 12 or π. But I want you to get practice reasoning without making those kinds of substitutions. The important point here is that $a, b_0, b_1, b_2, b_3, b_4$ are *not* numbers that change with x; they are constants.

End of notes.)

(a) Compute $T(a)$.

(b) Compute $T'(a)$.

(c) Compute $T''(a)$.

(d) Compute $T^{(3)}(a)$.

(e) Compute $T^{(4)}(a)$.

Exercise 16.1.3. Suppose somebody tells you they have a function $f(x)$, and that

- $f(0) = 1$.
- $f'(0) = 0$.
- $f''(0) = -1$.
- $f^{(3)}(0) = 0$.
- $f^{(4)}(0) = 1$.

(a) Can you find a degree four polynomial $T(x)$ such that

- $f(0) = T(0)$,
- $f'(0) = T'(0)$,
- $f''(0) = T''(0)$,
- $f^{(3)}(0) = T^{(3)}(0)$, and
- $f^{(4)}(0) = T^{(4)}(0)$?

(b) Would you expect the graphs of $T(x)$ and $f(x)$ to be related in any way? Why or why not?

16.1.1 The definition

Definition 16.1.4. Let f be a function, and choose a real number a. The *nth degree Taylor polynomial* of f at a is the degree n polynomial T_n satisfying

- $T(a) = f(a)$,
- $T'(a) = f'(a)$,
- $\ldots$,
- $T^{(n)}(a) = f^{(n)}(a)$.

In other words, T_n is the polynomial whose value, derivative, second derivative, $\ldots$, and nth derivative at a all agree with those of f at a.

Based on the previous page, we know that the nth degree Taylor polynomial can be written as

$$T_n(x) = b_0 + b_1(x-a) + \frac{b_2}{2}(x-a)^2 + \ldots + \frac{b_n}{n!}(x-a)^n$$

where $b_1 = f'(a)$, $b_2 = f''(a)$, $\ldots$, $b_n = f^{(n)}(a)$. For example, the coefficient in front of $(x-a)^4$ is given by $\frac{f^{(4)}(a)}{4!}$.

And, on the previous page, you were finding the 4th degree Taylor polynomial to some mystery function f.

Exercise 16.1.5 (If there is time)**.** Can you think of a trig function f that satisfies the conditions from the previous page? That is, so that

- $f(0) = 1$.
- $f'(0) = 0$.
- $f''(0) = -1$.
- $f^{(3)}(0) = 0$.
- $f^{(4)}(0) = 1$.

16.2 Example: cosine

Let $f(x) = \cos(x)$. We'll find the Taylor polynomials of f at $a = 0$. We'll also plot the graph of T_n next to the graph of $\cos(x)$ to compare.

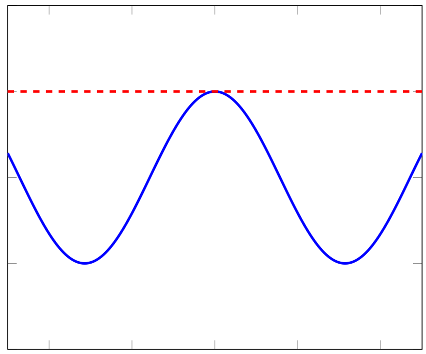
16.2.1 Degrees 0 and 1

Because $f(0) = 1$ and $f'(0) = 0$, we can find the degree 0 and degree 1 Taylor polynomials as follows:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1 + \cos'(0)(x-0) \\ &= 1 + 0(x-0) \\ &= 1 \end{aligned}$$

(Note that even though T_1 is called "degree one," it doesn't have a linear term, because the coefficient in front of $(x - a)$ turns out to be zero.)

Here is the graph of T_0 and T_1 (in dashed red) along with the graph of $\cos(x)$ (in solid blue):



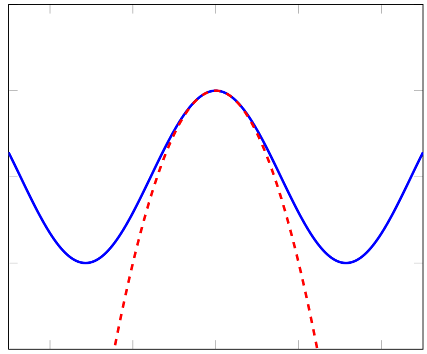
16.2.2 Degree 2 (and 3)

Because $f''(0) = -1$ and $f^{(3)}(0) = 0$, we can find the degree 2 and degree threeTaylor polynomials as follows:

$$\begin{aligned}
T_2(x) &= 1 + 0(x-0) + \frac{\cos''(0)}{2}(x-0)^2 \\
&= 1 + \frac{-1}{2}x^2 \\
T_3(x) &= 1 + \frac{-1}{2}x^2 + \frac{\cos'''(0)}{6}(x-0)^3 \\
&= 1 + \frac{-1}{2}x^2 + \frac{0}{6}(x-0)^3 \\
&= 1 + \frac{-1}{2}x^2
\end{aligned}$$

(Note that even though T_3 is called "degree three," it doesn't have a degree three term, because the coefficient in front of $(x-a)^3$ turns out to be zero.)

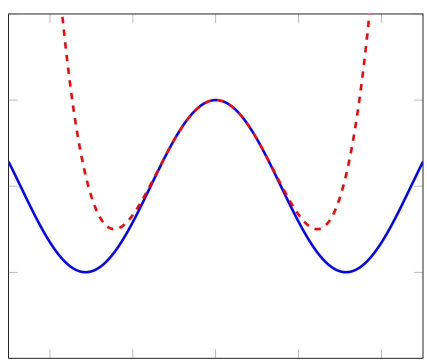
Here is the graph of $T_2(x)$ (which happens to be the same as $T_3(x)$ in this example) along with the graph of cosine:



16.2.3 Degree 4 (and 5)

$$\begin{aligned}
T_4(x) &= 1 + \frac{-1}{2}x^2 + \frac{\cos^{(4)}(0)}{24}(x-0)^4 \\
&= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 \\
T_5(x) &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 + 0x^5 \\
&= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4
\end{aligned}$$

Here is the graph of T_4 next to the graph of cosine:

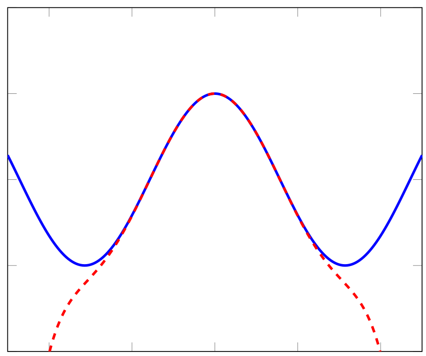


Are the graphs starting to look similar?

16.2.4 Degree 6

$$\begin{aligned} T_6(x) &= 1 + 0x + \frac{-1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 + 0x^5 - \frac{1}{720}x^6 \\ &= 1 + \frac{-1}{2}x^2 + \frac{1}{24}x^4 + \frac{-1}{720}x^6 \end{aligned}$$

Here is the graph of T_6 next to the graph of cosine:



The take-away: Taylor polynomials allow you to *approximate* a complicated function f by a simpler function (a polynomial), just by knowing the higher derivatives of f at a point a. As you can see from the graphs above, these approximations do a very good job *near* a. Further away from a, the polynomials may behave very differently from f.

16.3 Application: Approximating $\cos(0.5)$

Most of us do not know what $\cos(0.5)$ is off the top of our heads. But we saw in our previous drawings that the Taylor polynomials have graphs that are very similar to the graph of $\cos(x)$ when we are near $a = 0$.

So what if we evaluate $T_n(0)$? We get the following numbers:

- $T_0(0.5) = 1$
- $T_2(0.5) = 1 + \frac{-1}{2}(0.5)^2 = 0.875$
- $T_4(0.5) = 0.87760416667$
- $T_6(0.5) = 1 + \frac{-1}{2}(0.5)^2 + \frac{1}{24}(0.5)^4 + \frac{-1}{720}(0.5)^6 = 0.87758246528$

- $T_8(0.5) = 1 + \frac{-1}{2}(0.5)^2 + \frac{1}{24}(0.5)^4 + \frac{-1}{720}(0.5)^6 + \frac{1}{40320}(0.5)^8 = 0.87758256216$

Try comparing these to what your calculator says $\cos(0.5)$ is. I think you'll be pleased!

16.4 Preparation for next time

For next time, I expect you to be able to do the following.

Let $f(x) = \ln x$.

(a) Compute $f(1)$.

(b) Compute $f'(1)$.

(c) Compute $f''(1)$.

(d) Compute $f^{(3)}(1)$.

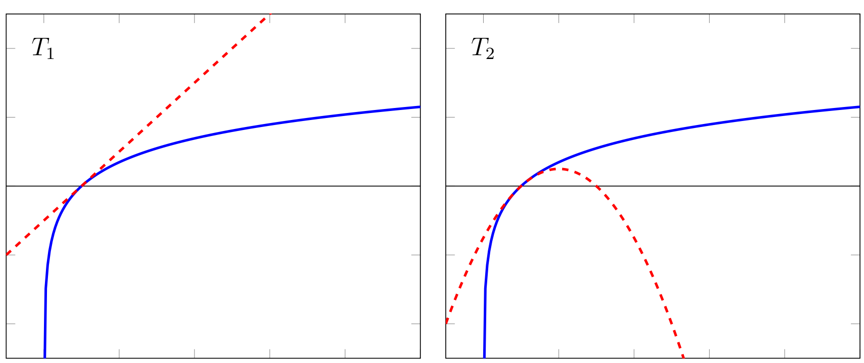
(e) Compute $f^{(4)}(1)$.

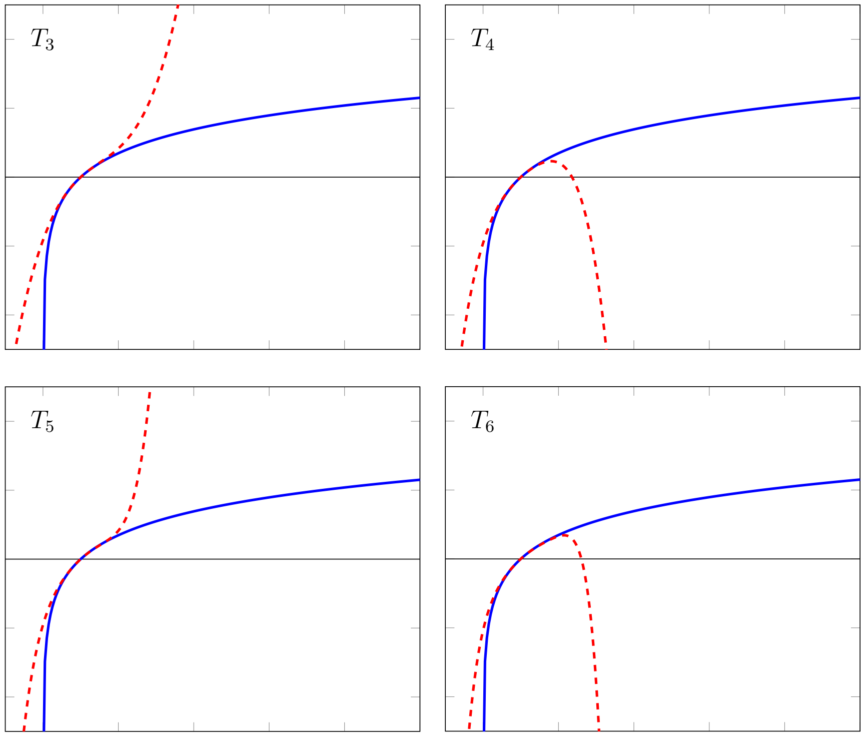
(f) Write the fourth degree Taylor polynomial $T_4(x)$ of $\ln x$ at $a = 1$.

To get you started (you will not be given this on the quiz), here is the degree 2 Taylor polynomial:

$$T_2(x) = 0 + (x-1) + \frac{-1}{2}(x-1)^2.$$

And, for fun, here are graphs of various Taylor polynomials for $\ln(x)$ at $a = 1$, graphed along with $\ln(x)$:





Also for fun: Check out this website to have fun with Taylor polynomials for different functions:

`https://www.geogebra.org/m/s9SkCsvC`.