Lecture 15

Logic. Differentiability.

15.1 If-then statements, and converses

In a past class we saw the following:

Proposition 15.1.1. Let x be a local extremum of f. Then x is a critical point of f.

The above proposition is using some new vocabulary. A *local extremum* is a point that is either a local minimum or maximum. (This means that the value of f at x is smaller than all nearby points' values, or larger than all nearby points' values.) And a critical point of f is a point x at which $f'(x) = 0$.

Another way to state the proposition is by using an if-then construction:

If *x is a local extremum of f*, then <u>x is a critical point of f.</u>

The italicized part is the *hypothesis* of the if-then statement, and the underlined part is the *conclusion* of the if-then statement.

I then asked in class whether if x is a critical point, then x must be a local extremum. We saw, in the example of $f(x) = x^3$, that this isn't true. For this function, $x = 0$ is a critical point, but $x = 0$ is not a local extremum. (By moving even a tiny bit to the right, f becomes larger, while moving even a tiny bit to the left, f becomes smaller.) In other words, the statement

"If *x is a critical point of f* then <u>x is a local extremum of f</u>"

is a **false** statement.

The two if-then statements above are very closely related. The two statements look identical, except the *hypothesis* and the conclusion have been swapped! We say that the two statements are **converses** to each other. (So the first statement is the converse of the second statement, and the second statement is the converse of the first statement.)

Remark 15.1.2 (Converse of, converse to)**.** The prepositions involved can be a little confusing. When "converse" is used as a noun, we often say "... is the converse *of* ...'

But when "converse" is used as an adjective, we often say "... is converse *to* ..."

What we have witnessed is: **A statement can be true even though its converse is false. A statement can be false even though its converse is true.**

Example 15.1.3. If a shape is a square, then it has four sides. (True statement.)

The converse statement is: If a shape has four sides, then it is a square. (False statement. There are four-sided shapes like parallelograms and rectangles that may not be squares!)

15.2 Differentiability

We've talked about the *Mean Value Theorem.* Like most mathematically meaningful statements, a theorem will be an "if-then" statement. (It will have a hypothesis and a conclusion.) But we need some new words to talk about the hypothesess, and to gain an appreciation for them.

Definition 15.2.1. Let f be a function, and x a number. We say that f *is differentiable at* x if the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists.

This limit should look familiar. It's the thing we write down to define and compute the derivative! I've mentioned a couple times that *when* the secant lines through x and $x+h$ converge to a single line, then we'll call the resulting single line is called the tangent line to f at x. And we'll call its slope the derivative.

But we haven't really looked at examples where the "when" fails. That is, we haven't looked at examples of functions where a tangent line might not exist at x!

Let's see such an example. It's pretty much the only example you'll need to know, at least for this course.

Example 15.2.2. Let $f(x) = |x|$.

What does the graph of this function look like? Well, remember that the absolute value function just returns the "size" of a number, or the distance of that number from 0. For example, $|5|$ is just 5. And $|-5|$ is also 5. Another way to think about absolute value is that it converts every number to its "positive form." So if a number is already positive, the absolute value just returns that number, while if a number is negative, the absolute value makes it positive.

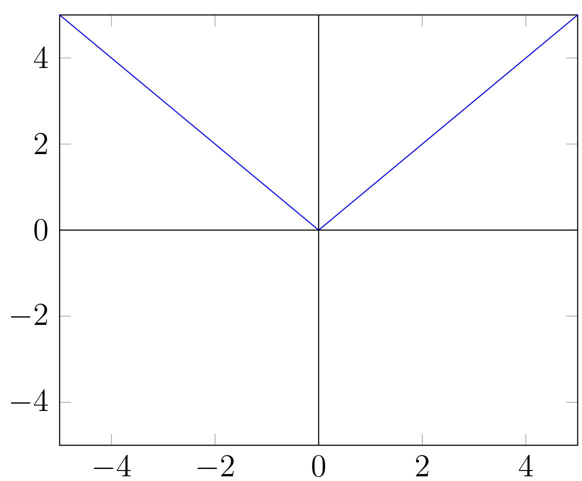
One way to express this is to use what's called a *piecewise* notation for the function:

$$f(x) = \begin{cases} x & x > 0 \\ -x & x < 0 \\ 0 & x = 0. \end{cases}$$

In other words, f sends a number x

- to itself if $x > 0$, or
- to negative of itself if $x < 0$, or
- to zero if $x = 0$.

Note that the graph of f will thus consist of two parts: The function $f(x) = x$ when x is positive, the function $f(x) = -x$ when x is negative, and the point telling us that $f(0) = 0$. All told, the graph (drawn in blue) looks like this:



I claim that $f(x) = |x|$ *does not have a tangent line at* $x = 0$.

Indeed, if you look at the secant line through x and $x+h$ for positive h, the secant line is a line of slope +1. But any secant line through x and $x+h$ for negative h is a line of slope -1.

So there's no way that a line of slope +1, and a line of slope -1, will converge to a single line as h approaches zero.

Put another way, if we have a function $g(h)$ that assigns +1 to h whenever $h > 0$, but assigns -1 whenever $h < 0$, then there's no value that $g(h)$ "wants to approach" as h approaches zero. So the difference quotient does not have a limit as $h \to 0$.

In sum, we see that

The function $f(x) = |x|$ is not differentiable at $x = 0$.

In other words, the function does not have a derivative at $x = 0$.

15.2.1 Other examples of non-differentiability

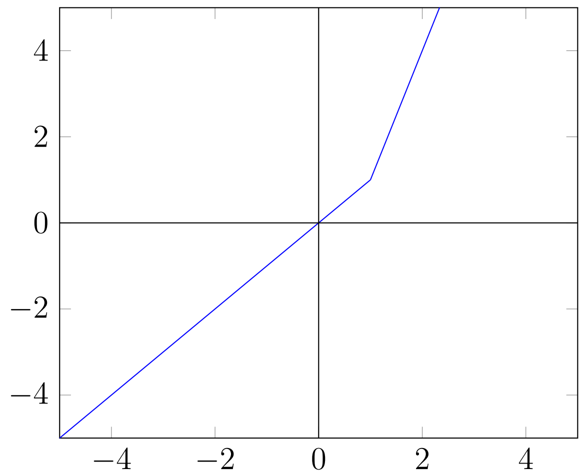
The absolute value function is not the only example of functions lacking a derivative somewhere.

In this class, we'll mostly deal with functions that have derivatives everywhere. However, it's important to know that there are functions that show up in real life that don't have derivatives.

And it's important to know one when we see one.

For example, the mean value theorem *relies* on the function in question being differentiable. (More precisely, it relies on the function having a derivative at every point along the interval (a, b).) So it's important to know that the conclusion of the mean value theorem can hold when the function isn't differentiable.

Example 15.2.3.



In the above graph of a function, there is a "kink" at $x = 1$ (where the height of the graph is equal to 1 as well). These "kinks" are tell-tale signs that a function is not differentiable at the kink.

Indeed, in this example, you can see that the function has a derivative everywhere else. This is because the graph of the function is modeled by parts of lines—a line to the left of the kink, and another line to the right of the kink. And we know that functions whose graphs are lines have derivatives everywhere.

Regardless, if you choose any $a < 1$ and $b > 1$, you will see that the conclusion of the mean value theorem fails to hold.

Example 15.2.4. The function below has a derivative everywhere except at the points $x = -3$, $x = -1$, $x = 1$, and $x = 3$.