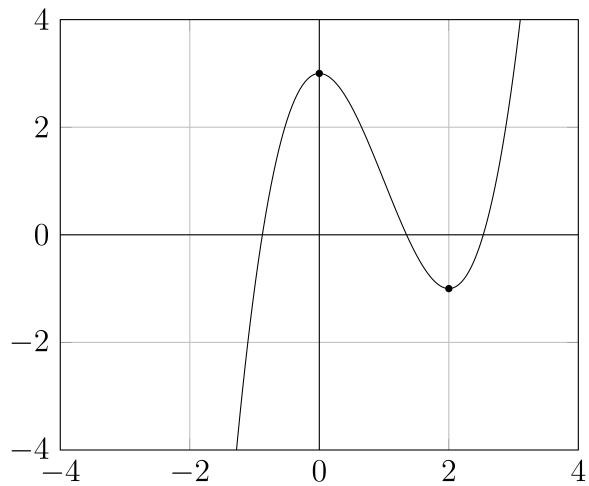
# Lecture 11

# Local extrema (minima and maxima)

Let's study the example of $f(x) = x^3 - 3x^2 + 3$:




Just by looking at the graph, we can see the two points where the derivative of $f$ is zero (i.e., the two points where the tangent lines are horizontal). They roughly occur at $x = 0$ and $x = 2$. (And you can prove that they *exactly* occur there if you do out the math—that is, if you solve the equation $f'(x) = 0$ for $x$.)

We see that at $x = 0$, the function is concave down. Moreover, it looks like $f(0)$ is the *biggest* value that $f$ achieves near $x = 0$. We will call such a point a **local maximum**. (That is, $x = 0$ is a local maximum.)

And at $x = 2$, we see that the function is concave up. Moreover, it looks like

$f(2)$ is the *smallest* value that $f$ achieves near $x = 2$. We call such a point a **local minimum** (so $x = 2$ is a local minimum). A point is called a local extremum (the plural is "local extrema") if it is either a local maximum or a local minimum.

**Remark 11.0.1.** Finding local maxima and minima have huge important in real life. For example, you could imagine $f(x)$ to measure a model for profit given a particular input $x$. Then you'd like to optimize to maximize profit within a feasible input range of values of $x$. In other words, you may want to find *local maxima* of the function $f$.

Likewise, when designing a particular system, $g(x)$ may measure the amount of risk, or the probability of failure, given some input parameter $x$. Then you would like to optimize to reduce risk as much as possible within some feasible input values of $x$. In other words, you may want to find *local minima* of the function $g$.

Your intuition might tell you that wherever there is a local maximum or a local minimum, the graph should have a "trough" or a "crest." In particular, the derivative should be zero there! This is true so long as the function is differentiable:

**Theorem 11.0.2.** If $f$ is a differentiable function, and if $x$ is a local minimum or a local maximum, then $f'(x) = 0$.

**Warning 11.0.3.** These minima and maxima are called "local." This is because if $x$ is a local minimum, it may not be true that $f(x)$ is the "minimum" value that $f$ can take!

In the example above of $f(x) = x^3 - 3x^2 + 3$, we see that $f(x)$ can take as negative a value as it wants, so $f$ has no "absolute minimum." Likewise, $f(x)$ can take as positive value as it wants, so $f$ has no "absolute maximum." It only has a "local" minimum at $x = 2$, where the value of $f(2)$ is smaller than the value at all *neighboring* points (i.e., all nearby points).

## 11.1 Critical points

So it will be important for us to find $x$ for which $f'$ vanishes. Such special points have a name:

**Definition 11.1.1.** Let $f$ be a function. We say that $x$ is a *critical point* of $f$ if $f'(x) = 0$.

**Example 11.1.2.** If $f(x) = 5$, every point is a critical point.

If $f(x) = 3x$, $f$ has no critical points.

If $f(x) = x^2$, $x = 0$ is a critical point.

In fact, zero is a critical point for $f(x) = x^3$ and for $f(x) = x^4$, and so forth.

**Warning 11.1.3.** Not all critical points are local extrema. (For example, look at the critical point of $f(x) = x^3$.)

**Warning 11.1.4.** If $f$ is not differentiable, not all local extrema are critical points. Consider the example of $f(x) = |x|$. This has a minimum at $x = 0$, but $f$ does not have a derivative there (as we have seen before).

## 11.2 The second derivative test

The following is called the **second derivative test** for finding local maxima and local minima. You in fact discovered it if you thought about Exercise 10.3.4.

**Theorem 11.2.1** (The second derivative test)**.** Suppose that $f'(x) = 0$ and $f''(x) > 0$. Then $f$ has a *local minimum* at $x$.

Suppose that $f'(x) = 0$ and $f''(x) < 0$. Then $f$ has a *local maximum* at $x$.

This helps us draw $f$: We know that $f$ looks like a hump/hilltop/crest where $f$ has a local maximum. And we know that $f$ looks like a bowl/trough/nadir where $f$ has a local minimum.

In simple terms, the theorem states the following: If the tangent to $f$ is flat at $x$, and $f$ looks like (part of) an upward-opening bowl at $x$, then $x$ must be the *bottom* of the bowl (hence a local minimum).

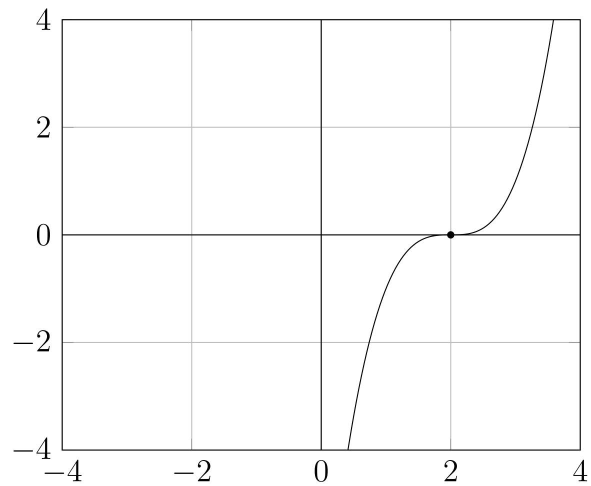
Likewise, if the tangent to $f$ is flat at $x$, and if $f$ looks like (part of) a downward-opening bowl at $x$, then $x$ must be the *top* of that bowl (hence a local maximum).

## 11.3 The second derivative test can be inconclusive

If $f'(x) = 0$ and $f''(x) = 0$, we *do not know* whether we have a local maximum or minimum (or neither)! Here are two examples:

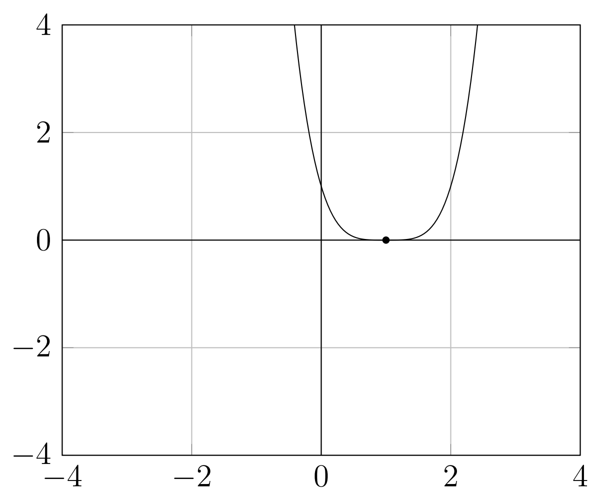
**Example 11.3.1.** Consider $f(x) = (x-2)^3$. Then—check this!—$f'(2) = 0$ and

$f''(2) = 0$. Below is a graph of $f(x)$:




This is a strange example, but it is a great one. As you can see, the graph does have "flat" tangent line at $x = 2$, but $x = 2$ is neither a local maximum nor a local minimum—I can immediately get larger than $f(2) = 0$ by moving right, or immediately get smaller than $f(2) = 0$ by moving left.

**Example 11.3.2.** Here is the example of $f(x) = (x-1)^4$. We can check easily that $f'(1) = 0$ and $f''(1) = 0$.




As we can see from the picture, we have a *local minimum* at $x = 1$.

The conclusion from the above two examples is: If the hypotheses of the second derivative test are not met, we have to do more work to determine whether we have a local minimum or maximum.

## 11.4 For next time

For next time you should be able to tell me the critical points, the local maxima, the local minima, and the the critical points where the second derivative test is inconclusive, for the following functions (and functions similar to them):

(a) $f(x) = x^3$

(b) $f(x) = x^3 - 3x^2 + 7$

(c) $f(x) = x \ln x$

(d) $f(x) = xe^x$.