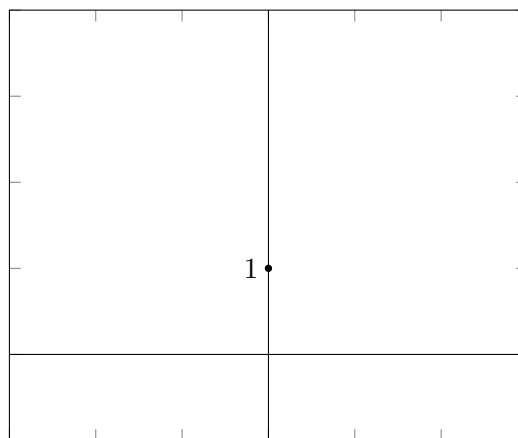
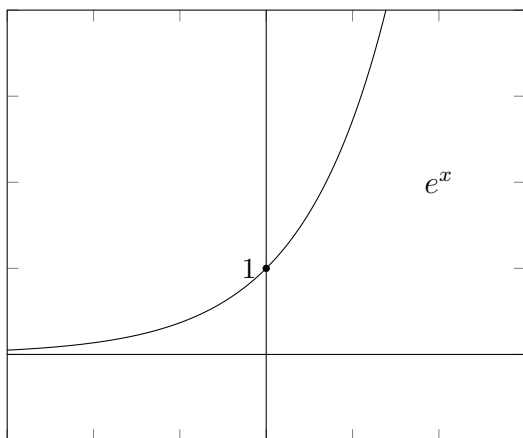


Lecture 7

Derivatives of exp and ln

7.1 Drawing derivatives of e^x

Exercise 7.1.1. Below on the left is the graph of $f(x) = e^x$.



Let me tell you the following fact: The derivative of e^x at $x = 0$ is 1. (In fact, the value of e^x at $x = 0$ is 1 also.)

- (a) Based on this, draw the derivative of e^x on the right.
- (b) How does your drawing compare to the graph of e^x ?

We haven't talked yet about how to draw the derivative of a function based on the graph of a function. Indeed, one can rarely draw the derivative "exactly," and

so our picture necessarily has to be a bit of a sketch. The only thing I want you to identify here is the following: As x becomes big and negative (so as x moves to the left) the tangent lines of the graph on the left become flatter and flatter. In other words, their slopes approach zero. So, the derivative of e^x approaches zero as x moves to the left.

On the other hand, if x becomes big and positive (so if x moves to the right) the graph of e^x looks steeper and steeper; so the derivative of e^x is getting bigger and bigger.

When you draw the graph of such a thing on the right, you begin to see that the graph of the derivative of e^x looks a lot like the graph of e^x itself.

7.2 e^x is its own derivative

In fact, we have the following theorem:

Theorem 7.2.1 (Derivative of e^x). The derivative of e^x is itself. That is,

$$(e^x)' = e^x.$$

Put another way,

$$\frac{d}{dx}(e^x) = e^x.$$

How cool is that? There's a function that is its *own* derivative!

Example 7.2.2. Let's find the derivative of e^{3x} . We have

$$\frac{d}{dx}(e^{3x}) = \frac{d}{dx}(3x) \cdot \frac{d(e^x)}{dx}(3x) \tag{7.2.1}$$

$$= 3 \cdot e^{3x}. \tag{7.2.2}$$

We have used the chain rule in the first line. If you're confused by it, it may be worthwhile to write this out step-by-step. Let's let $f(x) = e^x$ and $g(x) = 3x$. Then $e^{3x} = f(g(x))$. Thus

$$\frac{d}{dx}(e^{3x}) = \frac{d}{dx}f(g(x)) \tag{7.2.3}$$

$$= f'(g(x)) \cdot g'(x) \tag{7.2.4}$$

$$= f'(3x) \cdot g'(x). \tag{7.2.5}$$

(The second equality is due to the chain rule.) But we know that $f'(x) = e^x$ by Theorem 7.2.1, and we know $g'(x) = 3$ from previous lectures. Hence we can continue:

$$f'(3x) \cdot g'(x) = e^{3x} \cdot 3 = 3e^{3x}.$$

Exercise 7.2.3. Fix a real number B . Prove that the derivative of

$$f(x) = e^{Bx}$$

equals

$$Bf(x).$$

More generally, if you have another real number A , let

$$g(x) = Ae^{Bx}.$$

(For example, if you choose $A = 3$ and $B = 5$, you would have $3e^{5x}$. The previous example is when $A = 1$.) Prove that

$$g'(x) = Bg(x).$$

Application 7.2.4. This kind of behavior is incredibly important for *modeling*. For example, how fast is a population growing? In ideal circumstances, the more individuals there are in a population, the faster we expect the population to grow. Better yet, we might expect that the rate of population growth is *proportional* to the population itself! (Note that “being proportional to” is a far more precise relationship than “the bigger the population, the faster the growth”.)

That’s exactly what Exercise 7.2.5 tells us about $g(x) = Ae^{Bx}$. We see that g' is proportional to g (with proportional constant B). So for example, x could model time, while $g(x)$ could model the population at time x .

By the way, why might $g(x)$ be a *bad* model for population growth? For what kinds of situations might $g(x)$ be a *good model*? In those situations, what might A and B represent?

Exercise 7.2.5. Find the derivative of $f(x) = 5^x$. Hints: Remember that $5 = e^{\ln 5}$, remember the basic rules for dealing with exponents, and use the chain rule.

Exercise 7.2.6. Your friend is excited about the idea that $f(x)$ could equal $f'(x)$ and looks for more examples that look like e^x . They try $f(x) = 5^x$, and are disappointed that $f'(x)$ does not equal $f(x)$.

Is it possible to find any number k —other than e —so that if $f(x) = k^x$, then $f'(x) = f(x)$?

Remark 7.2.7. Isn’t e special?

Exercise 7.2.8. Now that you know the derivative of $g(x) = e^x$, can you figure out the derivative of $f(x) = \ln x$?

Hint: What is $g \circ f$? What if you try computing $(g \circ f)'$ using the chain rule, too?

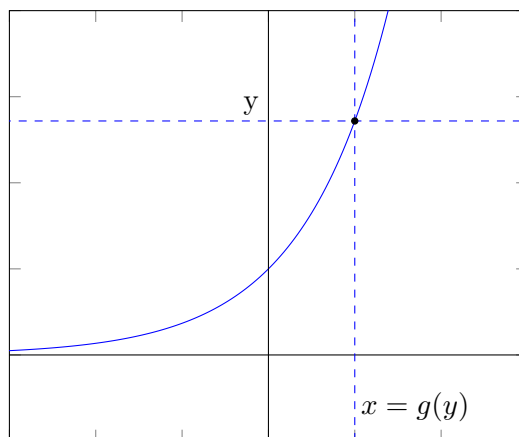
7.3 Review of right inverses

Let f be a function. Here's a question: Given a value of f , can we always determine which x it came from?

Example 7.3.1. Here are some examples:

1. If $f(x) = 3x$, and if someone tells you that f takes the value 12, you know exactly where: x must equal 4. In fact, in general, if f takes value y , you know the original x is $y/3$.
2. If $f(x) = 2^x$, and if someone tells you that f takes the value 8, you know exactly where: x must equal 3. In fact, in general, if f takes value y , you know f does so at $\log_2 y$.
3. If $f(x) = x^2$, and if someone tells you that f takes the value 4, you *don't* know exactly where: x could equal 2 or -2. However, *if* you restrict yourself to looking only for positive values of x , then if f takes value y , you know that the original x is \sqrt{y} .

Below is a visual way to think about this process. Drawn is the graph of f . Given a value y , can you figure out which value of x satisfies the equation $f(x) = y$? If so, that means that the coordinate x now becomes a function of y —you input y , and you output x —and we can call this function g .



Warning 7.3.2. While we were diligent about drawing g as a function of y before, from now on, we must now be comfortable realizing that letters are just letters, and we don't care if g takes inputs to be symbols that look like " x ," or symbols that look like " y "; that is, g will often be treated as a function of x , too.

Definition 7.3.3. Let f be a function. We say that a function g is a *left inverse* to f if

$$(g \circ f)(x) = x.$$

Put another way, g “remembers” the original value x that outputted $f(x)$.

We also say that f is a *right inverse* to g . Put another way, f “knows” that if $g(\text{something}) = x$, then $\text{something} = f(x)$.

7.4 Derivatives of right inverses

It turns out that if we know the derivatives of a function g , then—if g has a right inverse f —we can figure out the derivatives of the right inverse f .

Lemma 7.4.1. Let g be a function, and suppose that f is a right inverse to g , defined on some open interval containing x . Suppose also that g is differentiable at $f(x)$, and that $g'(f(x)) \neq 0$. Then

$$f'(x) = \frac{1}{g'(f(x))}. \quad (7.4.1)$$

That is, the derivative of f at x is computed by dividing 1 by the derivative of g at $f(x)$.

Proof. Let’s look at the following string of equalities:

$$1 = (x)' \quad (7.4.2)$$

$$= (g \circ f)'. \quad (7.4.3)$$

The first equality is our knowledge of the derivative of the function x . The next equality is using the hypothesis that g is a right inverse to f , so that $f \circ g = x$.

In total, what this string of equalities says is that the function on the righthand side is equal to the (constant!) function on the lefthand side. So let’s evaluate at some point x . We have

$$1 = (g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

We can divide both sides by $g'(f(x))$ so long as this number isn’t zero; so we find:

$$\frac{1}{g'(f(x))} = f'(x) \quad \text{when } g'(f(x)) \neq 0.$$

This is what we wanted. □

Example 7.4.2. $f = \ln(x)$ is a right inverse to $g(x) = e^x$. This is because

$$g \circ f(x) = e^{\ln x} = x.$$

We know the derivative of $g(x) = e^x$, so we can use the lemma to find the derivative of $f(x) = \ln(x)$! Let's try:

$$(\ln(x))' = f'(x) \tag{7.4.4}$$

$$= \frac{1}{g'(f(x))} \tag{7.4.5}$$

$$= \frac{1}{e^{f(x)}} \tag{7.4.6}$$

$$= \frac{1}{e^{\ln(x)}} \tag{7.4.7}$$

$$= \frac{1}{x}. \tag{7.4.8}$$

. The first equality is by definition of f . The next equality is using Lemma 7.4.1. The rest is just plugging in our knowledge of g' and \ln .

7.5 The derivative of natural log

The example from the last page is important, so let's record this as a theorem. (You will be expected to know this:)

Theorem 7.5.1 (The derivative of \ln). The derivative of \ln is “one over x .” That is,

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

7.6 For next lecture

You should be comfortable finding derivatives of functions involves e^x and \ln . For example, you should be able to find f' for each of the following functions f :

(a) $f(x) = e^x$

(b) $f(x) = e^{3x}$

(c) $f(x) = e^{3x+2}$

(d) $f(x) = 3e^x$

(e) $f(x) = 5^x$

(f) $f(x) = 5^{3x}$

(g) $f(x) = \ln(x)$

(h) $f(x) = \ln(3x)$

(i) $f(x) = \ln(x + 3)$