# Lecture 6

# The chain rule

Here are the summaries of the rules/laws we know for derivatives so far:

| Rule/Law | For derivatives |
|---|---|
| Constants | $(C)' = 0.$ |
| Scaling | $(mf)' = mf'$ |
| Sums | $(f' + g') = f' + g'$ |
| Powers | $(x^n)' = nx^{n-1}.$ |
| Composition | $(f \circ g)' =???$ |
| Products | $(fg)' =???$ |
| Quotients | $(f/g)' =???$ |

Today, we are going to practice taking derivatives of *compositions*. The rule we use to compute derivatives of composition is called the *chain rule*.

## 6.1 Review of compositions

The hardest part of applying the chain rule, for most calculus students, is actually understanding what a composition of functions is.

Remember that functions take inputs and produce outputs. A *composition* happens when a second function uses a first function's output as the second function's input. If you like, a composition is like a relay race in track and field. The first function passes a *number* onto the second function (instead of a baton).

When $f$ is a function, and $g$ is another function, we write

$$g \circ f$$

for the composition. When we evaluate $g \circ f$ at a number $x$, we have:

$$(g \circ f)(x) = g(f(x)).$$

The righthand side, in words, says: Apply $f$ to $x$, and whatever $f(x)$ is, plug it into $g$.

**Example 6.1.1.** Let $f(x) = x + 2$ and $g(x) = x^2$. Then

$$\begin{aligned}(g \circ f)(x) = &= g(f(x)) \\ &= g(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 2. \end{aligned} \tag{6.1.1}$$

**Example 6.1.2.** Let $f(x) = \sin(x)\cos(x)$ and $g(x) = x^2 + 3x + 2$. then

$$\begin{aligned}(g \circ f)(x) = &= g(f(x)) \\ &= g(\sin(x)\cos(x)) \\ &= (\sin(x)\cos(x))^2 + 3\sin(x)\cos(x) + 2. \end{aligned}$$

If you like, this last expression could also be written as

$$\sin(x)^2\cos(x)^2 + 3\sin(x)\cos(x) + 2 \qquad \text{or} \qquad \sin^2(x)\cos^2(x) + 3\sin(x)\cos(x) + 2$$

You can also try to compute the "outside function" first.

**Example 6.1.3.** Let $f(x) = \sin(x)\cos(x)$ and $g(x) = x^2 + 3x + 2$. then

$$\begin{aligned}(g \circ f)(x) = &= g(f(x)) \\ &= (f(x))^2 + 3f(x) + 2 \\ &= (\sin(x)\cos(x))^2 + 3\sin(x)\cos(x) + 2. \end{aligned}$$

## 6.2 The Chain Rule

**Theorem 6.2.1** (Chain rule)**.** Suppose that $g$ is differentiable at $x$, and that $f$ is differentiable at $g(x)$. Then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Put another way,

$$\frac{d(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$$

I want to emphasize in words what the chain rule says: If you want to compute the derivative of $f \circ g$ at $x$, then you must compute two things:

1. The derivative of $f$ **at g(x)**, and
2. The derivative of $g$ at $x$.

The product of these two numbers gives the derivative of $f \circ g$ at $x$.

Using the chain rule, you can find the derivative of functions like

(a) $(\sin(x))^3$

(b) $\sin(x^3)$

(c) $\cos(x^4 + 3x^3 - 2)$.

## 6.3 Identifying compositions

For many calculus students, the hardest part about taking derivatives is knowing whether we can use the chain rule in a particular situation. Why is this so hard? Well, in past classes, you've learned how to compose two functions, but you haven't learned to recognize whether a given function *arises* as a composition. Moreover, to use the chain rule, you need to be able to recognize the functions that are being composed.

**Example 6.3.1.** Let's write each of the functions below as a composition $g \circ f$. Importantly, let's identify the functions $g$ and $g$.

(a) $(\sin(x))^3$

(b) $\sin(x^3)$

(c) $\cos(x^4 + 3x^3 - 2)$.

Solution:

(a) What this expression tells us to do is to *first* evaluate $\sin(x)$, and then cube the result. So the first function is $f(x) = \sin(x)$ and the second, or "outside" function is $g(x) = x^3$.

(b) This expression tells us to first cube $x$, and then take sin of the result. So the first function is $f(x) = x^3$, and the second, or "outside" function is $g(x) = \sin(x)$.

(c) This expression tells us to take a number $x$, and first evaluate $x^4 + 3x^3 - 2$, and then take cos of the result. So $f(x) = x^4 + 3x^3 - 2$, and $g(x) = \cos(x)$.

You should check in all three examples that $(g \circ f)(x)$ indeed gives rise to the original expression.

## 6.4 Applying the chain rule

Now let's apply the chain rule.

**Example 6.4.1.** Find the derivative of $\sin(x^2 + 5)$.

**Solution.** We must first recognize that we behold a composition of two functions: On the outside is sin, while the inside is $x^2 + 5$. Hence we can use the chain rule.

$$\frac{d}{dx}(\sin(x^2+5)) = (\frac{d}{dx}\sin)(x^2+5) \cdot \frac{d}{dx}(x^2+5).$$

This is a product of two factors: The first factor, on the left, is the derivative of sin, evaluated at $x^2 + 5$. The second factor, on the right, is the derivative of $x^2 + 5$ (evaluated at $x$).

Because we know $\frac{d}{dx}\sin = cos$, and that $\frac{d}{dx}(x^2+5) = 2x$, we conclude:

$$\frac{d}{dx}(\sin(x^2+5)) = \cos(x^2+5) \cdot 2x.$$

Or, in more palatable notation,

$$\frac{d}{dx}(\sin(x^2+5)) = 2x\cos(x^2+5).$$

**Exercise 6.4.2.** Find the derivatives of the following functions:

1. $(\cos(x) + \sin(x))^3$
2. $\cos(\sin(x))$
3. $\cos(2x^4)$.

**Exercise 6.4.3.** We do not yet know how to take derivatives of a function like $h(x) = x^{1/3}$. *However*, we do know that if $g(x) = x^3$, then $g(h(x)) = x$.

Using this, and the chain rule, can you find a formula for $h'(x)$? That is, can you compute the derivative of $x^{1/3}$?

## 6.5 For next time: Exponentials, logarithms, and $e$ (A primer)

**Remark 6.5.1.** If you are already comfortable with functions like $e^x$ and $\ln x$, and how they relate to functions like $2^x$ and $\log_2 x$, you can focus on Section 6.5.6.

For next time, you'll need to be prepared to use exponentials and logarithms.

Consider the function $f(x) = 4^x$. You have seen this in precalculus. In fact, you probably knew that

$$4^0 = 1, \qquad 4^1 = 4, \qquad 4^2 = 4 \times 4 = 16, \qquad 4^3 = 4 \times 4 \times 4 = 64,$$

et cetera, back in high school. The cool fact is that even if $x$ is not an integer, $4^x$ is a number that makes sense.

**Example 6.5.2.** Here are some examples:

1. It makes sense to raise something to a negative power:

$$4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$

   More generally, we have that

$$4^{-n} = \frac{1}{4^n}.$$

2. It makes sense to raise something to a fraction:

$$4^{\frac{1}{3}} = \sqrt[3]{4} \text{ is the cube root of 4.}$$

   More generally, we have that

$$4^{1/n} = \sqrt[n]{4}$$

   is the $n$th root of 4. This root is the unique positive number so that its $n$th power is equal to 4.

3. Another fraction example:

$$4^{\frac{2}{3}} = \sqrt[3]{4^2}.$$

   (Note that this also equals $(\sqrt[3]{4})^2$.) Put into English, this means that $4^{2/3}$ is the cube root of $4^2$, or the square of the cube root of 4. More generally,

$$4^{\frac{a}{b}} = \sqrt[b]{4^a} = (\sqrt[b]{4})^a.$$

### 6.5.1 Exponent laws!

Let's see why the above statements are true.

You may have learned about the exponent laws in precalculus or back in high school. One of these laws says:

$$4^{a+b} = 4^a \cdot 4^b$$

That is, exponentiation takes "addition" to "multiplication." For example, $4^7 = 4^{2+5} = 4^2 \cdot 4^5$. Another law says:

$$4^{a \cdot b} = (4^a)^b = (4^b)^a.$$

This means that exponentiation takes "multiplication" to "powers." For example, $4^{21} = 4^{3 \cdot 7} = (4^3)^7$. Also, we have that $4^{21} = (4^7)^3$.

**Example 6.5.3.** Let's verify that the exponent laws are consistent with our knowledge of math. We have:

$$4^{2+3} = 4^5 = 4 \times 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4 \times 4) = 4^2 \times 4^3.$$

So indeed, $4^{2+3} = 4^2 \cdot 4^3$.

We also have:

$$4^{2 \cdot 3} = 4^6 = 4 \times 4 \times 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4) \times (4 \times 4) = (4 \times 4)^3 = (4^2)^3.$$

**Remark 6.5.4** (Reminder)**.** Let me also remind you that *anything* to the 0th power is equal to 1. For example, $5^0 = 1$. Likewise, $\pi^0 = 1$.

And, anything to the 1st power is that anything again. For example, $5^1 = 5$.

**Remark 6.5.5** (The reasoning for fractional and negative powers)**.** Knowing these laws is how you *create* the definitions for things like $4^{-3}$ and $4^{1/5}$. Indeed, if you know what $4^3$ is, and if you desire the law $4^{3+(-3)} = 4^3 \cdot 4^{-3}$ to be true, you *must* conclude that $4^{-3}$ is equal to $1/4^3$. For example,

$$4^3 \cdot 4^{-3} 4^{3+(-3)} = 4^0 = 1.$$

Dividing both sides by $4^3$, we see

$$4^{-3} = \frac{1}{4^3}.$$

Likewise, the other law of exponent tells us

$$5 = 5^1 = 5^{\frac{1}{2}\cdot 2} = (5^{\frac{1}{2}})^2.$$

Taking the square root of both sides, we find

$$\sqrt{5} = 5^{\frac{1}{2}}.$$

There's nothing special about the number 5 here; anything to the $\frac{1}{2}$ power is the square root of that anything. Likewise, anything to the $\frac{1}{3}$ power is the cube root.

### 6.5.2 The number $e$

The number $e$ is called Euler's constant sometimes, but it's usually just called $e$. (Eeee!) In your previous math classes, you probably didn't have too much reason to care about this deeply, except that it has some interesting roots in banking. However, you will see why $e$ is important in calculus.

For now, let me just say that $e$ is an irrational number, and here are the first few digits of its decimal expantion:

$$2.718281828459045235360287471352\ldots \tag{6.5.1}$$

We will soon be dealing with the function $f(x) = e^x$. You should think of this function as behaving very much like $f(x) = 4^x$. For example, we have that

$$e^0 = 1, \qquad e^1 = e, \qquad e^2 = e \times e \approx 7.38905609\ldots.$$

(We compute $e^2$ using a computer or calculator; if we have a lot of time at the end of this course, we'll see *how* a computer does this!)

### 6.5.3 The logarithm

The logarithm base $n$ of a number $x$ is written

$$\log_n x.$$

The number $\log_n x$ is the number you need to raise $n$ to in order to obtain a value $x$. For example,

$$\log_3 9 = 2.$$

This is because 2 is the number such that $3^2 = 9$. As another example,

$$\log_3 81 = 4.$$

(Just try computing $3^4$ to see why this is true.)

Put another way, it is always true that

$$3^{\log_3 x} = x.$$

We say that the logarithm base $n$ is the *inverse* function to exponentiation base $n$. (Put another way, if the output of the logarithm becomes the input of the exponential, the final output is the initial input.)

In fact, it is also true that

$$\log_3(3^x) = x.$$

**Exercise 6.5.6.** You should be able to compute the following:

(a) $\log_2 8$

(b) $\log_3 243$

(c) $\log_2 \sqrt{2}$

(d) $\log_\pi \pi^3$.

### 6.5.4 Natural logarithm

Because $e$ is so special[1], we give a special name to the logarithm base $e$. We define the *natural logarithm*, or the *natural log*, to be the logarithm base $e$, and we denote it as follows:

$$\ln$$

So for example,

$$\ln e = 1, \qquad \ln(e^3) = 3.$$

In general ln of a nice integer looks crazy; for example,

$$\ln 2 = 0.69314718056\ldots$$

so if you like integers and rational numbers, ln is not your best friend. But it will become a better friend as we realize how important ln and $e$ are in calculus—in fact, it is probably one of the most convincing pieces of evidence that crazy, *transcendental numbers* like $e$ have a place in our mathematical universe.[2]

[1] We will see why in the coming lectures

[2] There is another transcendental number, $\pi$, that is obviously very important to mathematics. If you don't know what a transcendental number is, don't worry; they are a special kind of irrational number.

In calculus, it will be very useful to know how to convert expressions like

$$5^{\text{some power}}$$

into expressions base $e$; that is, into expressions like

$$e^{\text{some other power}}.$$

**Example 6.5.7.** Let us convert $5^3$ into an exponent with base $e$. Here is our work:

$$\begin{aligned} 5^3 &= (e^{\ln 5})^3 && (6.5.2)\\ &= e^{(\ln 5)\cdot 3} && (6.5.3)\\ &= e^{3\ln 5}. && (6.5.4) \end{aligned}$$

The first equality is using the definition of logarithm base $e$. (Note that we don't need to know how to calculate $\ln 5$, but we know that it exists as a number, so we just use it.) The next equality follows from an exponent law: Exponentiation exchanges multiplication of powers with iterated powers. The last line is just re-writing the same expression in a nicer way.

### 6.5.5 Exponentials for non-rational powers

Now, you may not have thought deeply about how to calculate something like $4^x$ when $x$ is, say, an irrational number. In this class, you only need to know this *can* be done, and not *how* to do it.

So you only need to read this section if you're curious about how something like $4^\pi$ is computed. I'll illustrate by example.

**Example 6.5.8.** For example, how would you compute $4^\pi$? It's a three-step process.

First, we choose a collection of numbers that approximates $\pi$ really well. For example, we could choose

$$3, \qquad 3.1, \qquad 3.14, \qquad 3.141, \qquad 3.1415, \qquad 3.14159, \qquad \ldots$$

and so forth. Note that each of these numbers can be written as a fraction. (For example, 3.14 is equal to 314/100; you can simplify this fraction if you like.) In particular, we *know* how to calculate each of the following numbers:

$$4^3, \qquad 4^{3.1}, \qquad 4^{3.14}, \qquad 4^{3.141}, \qquad 4^{3.1415}, \qquad 4^{3.14159}, \qquad \ldots$$

and so forth.

Calculating all these numbers is the second step. For your edification, here are the answers:

$$64, \qquad 73.516\ldots, \qquad 77.708\ldots, \qquad 77.816\ldots, \qquad 77.870\ldots, \qquad 77.880\ldots,$$

and so forth.

Now, here is the third and most fun/difficult step. We have to *prove* that this collection of numbers "converges" to some number—put another way, that this sequence has a limit.[3] Then we *define* $4^\pi$ to be this limit.

### 6.5.6 For the quiz!

For the quiz, you should be able to simplify the following expression:

(a) $e^{\ln 3}$

(b) $e^{\ln e}$

(c) $e^{\ln x}$

(d) $e^{\ln 1}$

(e) $e^{\ln \pi}$

(f) $\ln(e^3)$

(g) $\ln(e^\pi)$

(h) $\ln(e^x)$

(i) $\ln(e^3 \cdot e^5)$

(j) $\ln(e^3 \cdot e^x)$

(k) $\ln(e^3 \cdot e^{-3})$

[3]Warning: This notion of limit is slightly different from the notion of limit we have discussed before. This is the limit of a *sequence of numbers*, while we have discussed in this class the limit of a *function* at a point.