Lecture 4

Derivatives of polynomials

We have seen different ways to talk about the derivative:

Definition 4.0.1. Let f be a function, and choose a number x.

(a) (Using geometry) If f has a tangent line[1] at x, then the slope of that tangent line is the derivative of f at x.

(b) (Using algebra) If the expression

$$\frac{f(x+h)-f(x)}{h}$$

approaches a number as h approaches 0, then that number is the derivative of f at x.

(c) (Using algebra, and with limit notation) The derivative of f at x is

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(if this limit exists).

These are all definitions of the derivative. But because the derivative is something we'll be using so much, we're going to want some briefer notation.

[1]Remember that a "tangent line" is the line that the secant lines through P and Q approach, as Q nears P. Here, $P=(x,f(x))$.

4.1 Notations for the derivative

Here are the most common notations used for derivatives:

Notation 4.1.1. Let f be a function and x a number. Then the derivative at x is denoted by any of the following:

- $f'(x)$
- $\frac{df}{dx}(x)$
- $\left(\frac{d}{dx}f\right)(x)$.

The above notations all mean the same thing. The notation we'll use most often is $f'(x)$. This is read "f **prime of** x**.**"

Example 4.1.2. In lab you saw that if $f(x) = 3x + 2$, then the derivative of f at 2 is given by 3. In other words,

$$f'(2) = 3.$$

You also saw in lab that if $g(x) = 5x^2$, then the derivative of g at 2 is given by 20. So

$$g'(2) = 20.$$

(That is, "g prime of 2 is 20.") You could also have written

$$\frac{dg}{dx}(2) = 20.$$

4.2 The derivative as a function

In lab, you probably plugged in a value for x, and then evaluated the difference quotient. For example, for $g(x) = 5x^2$, to evaluate the derivative at $x = 2$, you probably began by writing out the fraction

$$\frac{g(2+h) - g(2)}{h} = \frac{5(2+h)^2 - 5(2)^2}{h}.$$

But one think you could also have done, is to *not* plug in $x = 2$, and just leave it as x. It may feel a little strange to have two variables (both x and h) floating around, but we can still treat them as numbers and perform our calculations:

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{5(x+h)^2-5x^2}{h}\\ &= \frac{5x^2+10xh+5h^2-5x^2}{h}\\ &= \frac{10xh+5h^2}{h}.\end{aligned}$$

This final expression, as usual, simplifies when $h \neq 0$ (i.e., when h does not equal zero):

$$10x + 5h.$$

Then, as h approaches zero, the expression $10x + 5h$ becomes $10x$. So we conclude:

The derivative of g at x is given by $10x$.

Or, using the "prime" notation we just learned, we can write:

$$g'(x) = 10x.$$

But wait! This g' is a new function!

What do I mean? Remember that a function is something that eats a number, and spits out another number. For example, when we write $f(x) = 3x$, we mean that f is a function that takes in a number, then spits out three times that number. Likewise, $g(x) = 5x^2$ is a function that has you input a number (called x) and that squares it, then multiplies it by five ($5x^2$).

Likewise, we see that g' is something that will take in a number (called x) and output a new number (called $10x$).

Upshot. If f is a function, then f' is a new function.

Terminology. If f is a function and x is a number, then the number $f'(x)$ is called the *derivative of f at x.* On the other hand, the function f' is called *the derivative of f.*

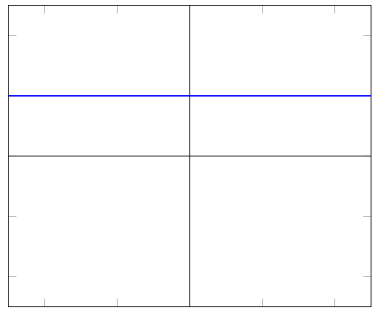
Today we'll study what f' looks like when f is a polynomial. But first, some basic laws about derivatives:

4.3 Basic derivative laws

Our ultimate goal in life (just kidding—our goal in this class) is to take a function f, and to be able to understand the new function f'. That is, to understand the derivative of f. Here are some of the most basic laws about how to write f' given f:

4.3.1 Constant functions have zero derivative

Suppose $f(x) = 3$. This is called a "constant function," because no matter what the input number is, f constantly outputs 3. The graph of a constant function is a horizontal line.



(Above, in blue, is an image showing the graph of $f(x) = 3$.) Of course, this line itself has slope zero. In fact, any secant line to f is just the graph of f itself! Accordingly, the tangent line to f is f itself. So f' is another constant function, with value zero.

Law. *If f is a constant function, then $f' = 0$.* That is, regardless of x, $f'(x) = 0$.

Example 4.3.1. If f is the constant function given by $f(x) = \pi$, then f' is the function given by $f'(x) = 0$.

Remark 4.3.2. We can think about this physically, too. If $f(t)$ represents the positive of someone at time t, then f being constant means that person isn't moving. So the speed of that person is always zero—that is, the derivative of f is zero.

Remember that we saw that slopes of lines had to do with speed, when looking at a position-versus-time graph for motion with constant speed. What we're going to be using from hereon is that the slope of the *tangent line* of a position-versus-time graph (regardless of whether speed is constant) is the speed that a (very accurate) speedometer would show at that time.

4.3.2 Scale a function, scale the derivative

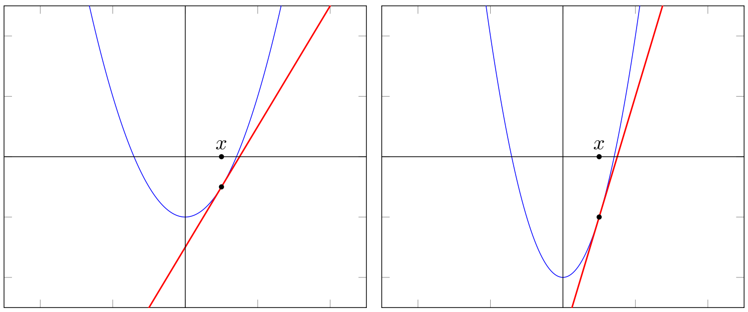
Suppose that f is some function. How is the derivative of f related to the derivative of $5f$? For example, how is the derivative of x^3 related to the derivative of $5x^3$?

Law. $(af)' = a(f')$ (for any number a).

Let's parse this law. It means that if I take a function, multiply it by a, and *then* take the derivative, I'll get the same answer as first taking the derivative, then multiplying by a.

Example 4.3.3. Suppose $f(x) = 5x^2$. We saw earlier that f' is the function given by $f'(x) = 10x$. If we let $g(x) = 10x^2 = 2f(x)$, then $g' = 2f'$, so $g'(x) = 2f'(x) = 20x$.

Sometimes, multiplying by a number a is called "scaling by a." So this law says that if you scale a function, you scale its derivative.



Above, on the left is a picture of the graph of a function $f(x) = x^2 - 2$, together with a tangent line at $x = 1$, in red. On the right is the picture of a graph of a function $2f$, that is, $2x^2 - 4$, together with a tangent line at $x = 1$ (the same x as for the lefthand picture). Though it's not obvious from the pictures, the slope of the tangent line on the right is *twice* the slope of the tangent line on the left. In other words, the slope of the tangent line is scaled by the same factor by which the function was scaled.

Remark 4.3.4. We can again think about this physically. Suppose $f(t)$ is a function that again tells you your position of at time t. What would it mean to scale f by a number a? Well, it means that your position is a times as far at any given time. (For example, if $a = 3$, then your position would be triple what it would have been with the unscaled, original function.) What the scaling law says is that if you're always ending up three times as far, you're always moving three times as fast.

4.3.3 Add functions, add derivatives

Law. $(f + g)' = f' + g'$.

That is, if you add two functions, and then take their derivative, you'll get the same answer as taking the derivative of each function, and then adding them.

Remark 4.3.5. This is a natural-looking law, but it's one of the harder ones to justify using pure geometry. It's easier to justify if you think physically.

Suppose that $f(t)$ represents the position of a train at time t.[2] This could be measured, for example, by someone outside the train, observing the train.

Suppose further that there is a cheetah inside the train, and $g(t)$ represents the position of the cheetah "relative to the train" at time t. This is measured, for example, by somebody *inside* the train, observing the cheetah. [3]

The the function $f+g$, which at time t outputs $f(t)+g(t)$, represents the position of the cheetah.[4] This could be measured, for example, by somebody outside the train, with x-ray vision, observing the cheetah.

So how are the speed of the train, the speed of the cheetah relative to the train, and the actual speed of the cheetah, all related?

Well, the actual speed of the cheetah would be the *sum* of the speed of the train with how fast the cheetah is moving relative to the train![5]

Remember that the slope of a line has to do with speed. We saw this for position-versus-time graphs of objects moving with constant speed. And if our position-versus-time graph is a curve, then we can interpret the *derivative* (which is the slope of the tangent line) as the speed that a (highly accurate) speedometer would show at the time.

So this physical "thought experiment" displays one physical argument why the derivative of a sum of functions should be the sum of their derivatives.

4.4 Derivatives of powers

The next law, however, is one that I have never found a good physical or visual argument for. The only reason it's true—that I know of—is algebra.

Law. (The power law.) Suppose $f(x) = x^n$ where n is any number that's not zero. Then

$$f'(x) = nx^{n-1}.$$

[2]For example, it could represent how far the center of the train has moved along the track from a departure station.

[3]To make things even more concrete, $g(t)$ could represent how far away the cheetah is from the center of the train, at time t. In any case, if the cheetah is just seated in its passenger seat, $g(t)$ would be constant.

[4]For example, how far the cheetah is from the departure station.

[5]For example, if the train is moving 60 miles per hour to somebody watching the train from outside, but if the cheetah is just sitting still inside the train, then the outside observer with x-ray vision would perceive the cheetah to be moving at 60 miles per hour. On the other hand, if an inside-the-train observer saw the cheetah running 60 miles per hour in the direction opposite the train's front, then to an outside observer with x-ray vision, the cheetah would look stationary, as though it were running on a treadmill.

Example 4.4.1 ($n = 1$)**.** If $f(x) = x$, then $f'(x) = 1$.

Example 4.4.2 ($n = 2$)**.** If $f(x) = x^2$, then $f'(x) = 2x$.

Example 4.4.3 ($n = 3$)**.** If $f(x) = x^3$, then $f'(x) = 3x^2$.

Example 4.4.4 ($n = 4$)**.** If $f(x) = x^4$, then $f'(x) = 4x^3$.

Example 4.4.5 ($n = 5$)**.** If $f(x) = x^5$, then $f'(x) = 5x^4$.

This is a pattern that you'll have to get used to. You will see it *all the time* in this class. Make sure you know the power law!

4.5 For next time

For next time, you'll want to know how to take the derivative of a polynomial.

What is a polynomial? Here are examples:

- 1
- 3
- πx
- $x + 2$
- $9x^2 + \frac{1}{2}x$
- $\pi x^2 + 2x + 9$
- $3x^3 + \sin(1)x^2 + \cos(\pi)x + \pi^3$.

The key thing to note about all the above expressions is that they all look like

$$ax^3 + bx^2 + cx + d$$

where a, b, c, d are arbitrary numbers. (They could be zero, they could be π^3, whatever!) In fact, there's no limit to the power of x in a polynomial—another example is something like

$$x^{10,003} + x^2 + 1.$$

The important thing is that each time the symbol x shows up, the exponent of x is some integer that's not negative, and that x isn't inside some other function like sin, or e^x, et cetera.

Remark 4.5.1. In principle, polynomials are supposed to be the "simplest" kinds of functions. For example, if $f(x) = 3x^3 + 2x^2 + x - 9$, you would be able to tell me things like $f(3)$ and $f(-2)$ by hand. You just need to multiply and add a lot.[6]

They key thing to note is that a polynomial function is always a (i) sum of (ii) scaled versions of (iii) x^n for some n.

And the derivative rules from today show us how to deal with sums, with scaling, and with "power" functions like x^n. So we actually now know how to take derivatives of polynomial functions!

Example 4.5.2. Find the derivative of $f(x) = 3x^2 + x - 9$.

Answer. f is a sum of three terms: $3x^2$, x, and -9. So let's try to take the derivative of each.

The derivative of -9 is easy. This is a constant function, so its derivative is zero.

The derivative of x is also easy. For example, the graph of $g(x) = x$ is just a line with slope 1, so its derivative is always 1. You could also have used the power law if you wanted, because $x = x^1$.

Finally, $3x^2$ is 3 times x^2. We know that the derivative of x^2 is given by $2x$ by the power law. So, scaling this by 3, we conclude that the derivative of $3x^2$ is given by $6x$.

Putting this all together:

$$
\begin{aligned}
(3x^2 + x - 9)' &= (3x^2)' + (x)' + (-9)' \\
&= 3 \cdot (x^2)' + 1 + 0 \\
&= 3 \cdot 2x + 1 + 0 \\
&= 6x + 1 + 0 \\
&= 6x + 1. \qquad (4.5.1)
\end{aligned}
$$

And that's it!

Example 4.5.3. Find the derivative of $f(x) = x^5 - 7x^4 + 2x + 13$.

I'll tell you that the answer is $5x^4 - 28x^3 + 2$. Can you figure out why?

For next time, you should be able to take derivatives of polynomials. You'll get plenty of practice in lab.

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[6]On the other hand, if $f(x) = \sin(x)$, you probably would not be able to tell me, by hand, what $\sin(3)$ is. You would need a calculator! We'll see later in this class, if we're lucky, how you might able to calculate $\sin(3)$ to many decimal places without a calculator.