Lecture 3

Derivatives

Last time we drew some pictures to define what a tangent line is. The goal of today is to convert this *geometry* (which involves pictures) into *algebra* (which involves formulas).

Recap. Let f be a function and let P be a point on the graph of the function. Then for any other point Q on the graph, the line through P and Q is called a *secant line.*

Suppose that, as you repeatedly re-choose Q to be closer and closer to P, the secant lines approach a single line. This line is called the *tangent line at P to the graph of f.*

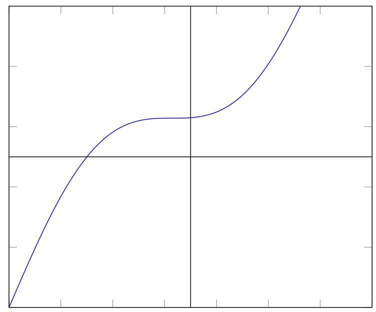
Warning. It might be hard to imagine, but sometimes, the secant lines we draw do *not* approach a single line as Q approaches P. In such a situation, we say that there is *no* tangent line at P.

Today. If the tangent line exists, we will define the *derivative* of f at P to be the slope of the tangent line.

Today we'll give the definition of the derivative using a formula, but it will be a little abstract. It will contain something called a "limit," which we will not formally study until the end of the semester. So we will leave the "limit" as part of the car that we won't study just yet. Starting next week, we'll see how to actually compute the derivative (so we can start driving).

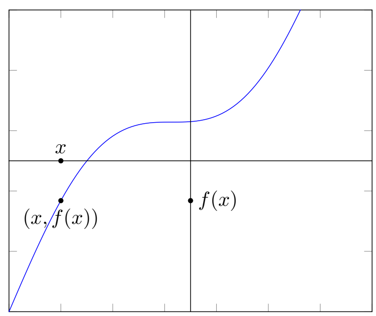
3.1 Points on graphs of functions

Below is the graph of a function f.



As you may have learned in a previous class, a choice of a number, x, determines a point on the graph! The point is determined by moving x units along the x-axis, and then moving $f(x)$ units vertically. The symbol we use for this point is:

$$(x, f(x)).$$



In the above picture, x is some negative number, and $f(x)$ happens also to be some negative number. Moreover, because f is a function (so its graph passes the vertical line test) if P is a point on the graph of f, we may write the coordinates of P as

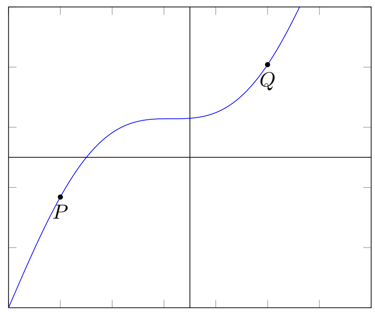
$$P = (x, f(x))$$

for some number x.

Because a number x determines a point P on the graph of f, and because the point P determines x, we will often talk about **how a function behaves "at x"** instead of how the function behaves at P.

3.2 Slopes of secant lines

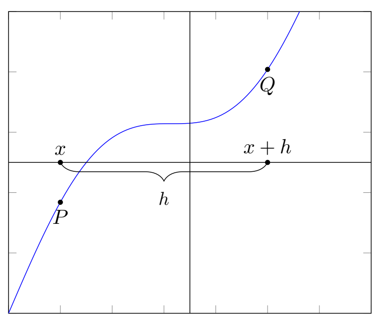
Now, let's say somebody chooses another point Q on the graph of f.



We may ask about the *horizontal* difference between P and Q—that is, what is the difference between the x-coordinate of P, and the x-coordinate of Q? Whatever it is, let us call it h:

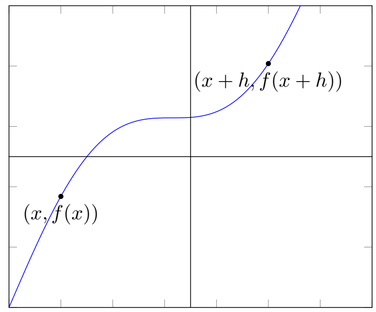
$$h = \text{The difference between the } x\text{-coordinates of } P \text{ and } Q.$$

So that the x-coordinate of Q is given by $x + h$.



(Note that h could be a positive or a negative number. In our pictures, h happens to be positive.) Then the coordinates of Q are given by:

$$Q = (x + h, f(x + h)).$$



Proposition 3.2.1. The slope of the secant line through P and Q is given by the formula

$$\frac{f(x+h)-f(x)}{h}.$$

Remark 3.2.2. In these notes, you will see me label sections by "Remarks" or "Propositions." (There will be other labels you'll see as the class goes on.)

A "Remark" is just a comment I would like to make.

A "Proposition" is a term used throughout mathematics. A Proposition is a statement that is true, and useful for getting an idea for what's going on, and also not too difficult to convince somebody of. For many students, you won't lose too much sleep if you replace the word "Proposition" by the word "Fact," but it is also very healthy to wonder *why* certain facts are true. If the fact is a Proposition, I promise you will be able to see why the fact is true if you spend a reasonable amount of time thinking it through.

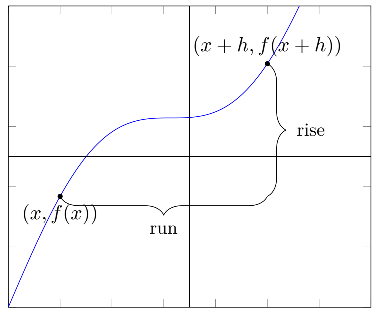
A "Proof" is a series of sentences of equations to convince you that the Proposition is true.

Proof. The slope of the secant line is given by "rise over run." The rise is the difference in the y-coordinates of P and Q, so

$$\text{rise} = f(x+h) - f(x).$$

The run is the difference in the x-coordinates of P and Q, so

$$\text{run} = (x+h) - x = h.$$



Then

$$\frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}.$$

□

Remark 3.2.3. The above proposition is useful because it takes a *geometric* idea (like drawing lines through points of graphs) and converts it into *algebra* (e.g., a formula).

For example, even if you had no idea what the graph of f looked like, if somebody gives you a formula for f, you can now compute the slope of a secant line. This is very powerful. It takes a lot more time to try to visually draw and measure something, than to just compute it using algebra.

Because we'll see this fraction a lot, we give it a name: $\frac{f(x+h)-f(x)}{h}$ is called a *difference quotient*. You became familiar with this expression for today's lecture.

3.3 The slopes of tangent lines

So let's choose a function f and a point P on the graph of f.

Remember that the tangent line to f at P is the line that the secant lines through P and Q approach as Q gets closer and closer to P. Well, if a bunch of lines approach a single line, then it stands to reason that the slopes of the those lines approach the slope of the single line.

Well, how can we talk about what it means for "Q to approach P"? Remember that, earlier, we chose h to be the horizontal difference between P and Q. So if the point Q is approaching P, then surely h is shrinking to zero!

On the other hand, we saw that the fraction $\frac{f(x+h)-f(x)}{h}$ is the slope of the secant line between P and Q.

So what we need to understand is the following:

Question 3.3.1. How does the number $\frac{f(x+h)-f(x)}{h}$ behave as h becomes closer and closer to zero?

Warning 3.3.2. Note that the fraction above has h in the denominator. In other words, *you cannot just plug in* $h = 0$ into the fraction. (One of the golden rules of mathematics is: You cannot divide by zero.)

3.4 The derivative (using words)

This question above (Question 3.3.1) is at the heart of calculus.[1]

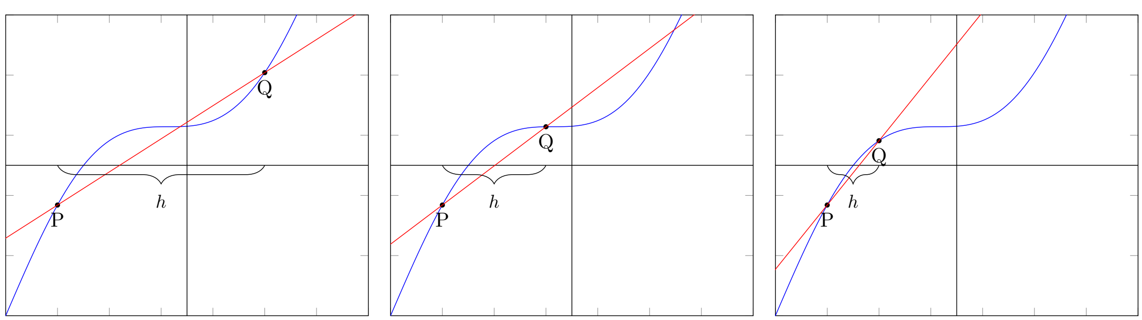
Definition 3.4.1. Let f be a function, and x a number. Then *the derivative of f at x* is the number that $\frac{f(x+h)-f(x)}{h}$ approaches as h gets closer and closer to zero (**if such a number exists**).

Remark 3.4.2 (What is a definition?)**.** A *definition* is, as you know, something you usually find in a dictionary. In a set of math notes, or in a math textbook, a definition is like a shortcut.

For example, the above definition of a derivative is very wordy! Too many words. So instead of saying all that ("the number that this fraction approaches as h goes to zero," if it exists) we get to just say "the derivative." Isn't that a lot shorter?

For example, I could say "Erica is my friend," rather than saying "Erica is someone with whom I have a good relationship and with whom I sometimes hang out." If a mathematician were to introduce the word *friend* in a textbook, they might write: **Definition.** Let P be a person. Then P is called a *friend* if P is a person with whom you have a good relationship and with whom you sometimes hang out.

Just as a visual reminder, here is what's geometrically happening as "h approaches zero":



[1] Really, the heart of half of calculus. The other half of calculus is devoted to integrals, which we'll see later on this semester.

Because h is the difference in the x-coordinates of P and Q, as h approaches zero, P and Q are getting closer and closer. And remember, we are not moving P around; we're just letting Q move. So "as h approaches zero," is a way to say "as Q moves closer to P."

3.5 The derivative (using limit notation)

Our definition of limit is a mouthful. That's why we also invent notation to make things easier. For example, $f(x) = 5x^3 + 3x + 1$ if far easier to see than "let f be a function that takes a number, cubes it, multiplies the result by 5, then takes the original number, multiplies it by 3, then adds that, then finally adds 1 to all that previous stuff." So here is the notation we will be using in this class:

Definition 3.5.1 (The derivative, using limit notation)**.** Let f be a function and x a number. Then the *derivative of f at x* is the number

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if it exists.

This notation is read "the limit as h approaches 0 of $\frac{f(x+h)-f(x)}{h}$." This is just notation for now, and you're not expected to have a deep understanding of this notation yet. We'll get more intimate with this notation later in the semester.

3.6 Example

Let's choose $f(x) = x^2 + 10$, and $x = 3$. Then

$$\begin{aligned} \frac{f(3+h) - f(3)}{h} = \frac{((3+h)^2 + 10) - (3^2 + 10)}{h} &= \frac{3^2 + 2 \cdot 3h + h^2 + 10 - 3^2 - 10}{h} \\ &= \frac{6h + h^2}{h}. \end{aligned}$$

Now, the question is, how does this fraction behave as h approaches zero? Here's the thing: Whenever h does not equal zero, the above fraction can be simplified:

$$\frac{6h + h^2}{h} = \frac{6h}{h} + \frac{h^2}{h} = 6 + h.$$

So, now we ask:

Question. What does $6+h$ become as h approaches zero?

Well, as h gets smaller and smaller, $6+h$ becomes a number closer and closer to 6. So the answer to the question is: $6+h$ becomes 6 as h approaches zero.

This is our first derivative that we've ever computed! The derivative of $f(x) = x^2+10$ at $x=3$ is given by 6.

3.6.1 Summary

Whenever you've solved a problem, it's good to look back on what you did.

1. We first wrote out the difference quotient $\frac{f(x+h)-f(x)}{h}$ and plugged in $x=3$.

2. We simplified it as far we could, keeping in mind that we *should not* divide by h. We ended up with $\frac{6h+h^2}{h}$.

3. We then tried to understand the behavior of the fraction when h does not equal zero. This is a great thing to do, because when h does not equal zero, we *can* simplify the fraction when the numerator has h in every term. We ended up with having to understand how $6+h$ behaves as h equals zero.

4. Well, we can reason out that as h approaches zero, $6+h$ approaches 6. And that's our answer.

The notation we can use for this is:

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = 6$$

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3.7 Preparation for next lecture

For each of the following functions f, you should be able to compute the difference quotient, and compute what the difference quotient approaches as h approaches 0.

(a) $f(x) = x^2$.

(b) $f(x) = x^2+5$.

(c) $f(x) = x^3$.

(d) $f(x) = x^3 + 7$.

Example 3.7.1. If $f(x) = x^3 + 6$, then

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{((x+h)^3 + 6) - (x^3 + 6)}{h} \\
&= \frac{(x^3 + 3x^2h + 3xh^2 + h^3 + 6) - (x^3 + 6)}{h} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 6 - x^3 - 6}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h}. \\
&= \frac{3x^2h}{h} + \frac{3xh^2}{h} + \frac{h^3}{h}.
\end{aligned}$$

When h does not equal zero, this simplifies to:

$$3x^2 + 3xh + h^2.$$

As h approaches 0, this expression becomes $3x^2$.