Lecture 1

Slopes and Introductions

Welcome to class! See syllabus for detailed logistical information. For now, you should know:

1. You are expected to come to every class and every lab via Zoom. Groupwork and discussion is a large part of your learning for this class.

2. You should let Hiro know as soon as possible if you foresee any technical accessibility issues. (For example, is your internet always slow? Do you only have a smartphone—and no tablet or laptop—so that the lecture videos will always be too small?)

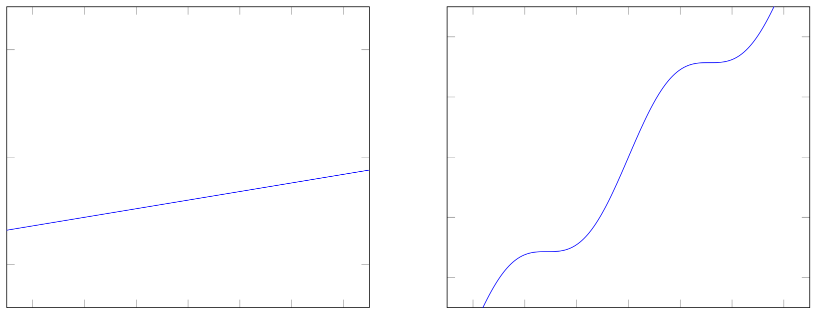
3. Every night of lecture, you will have a small quiz. What will be on the quiz? See the section "For next lecture" at the end of each day's class notes. The quiz will be due every day of class by 11:59 PM of that day.

x

1.1 Lines and slopes

Question: What's the difference between curves and lines?

In everyday life—if you're not a mathematician, nor in a math class—we may not think of any difference between a curve and a line. But as a mathematical term, a line is always a *straight line*, and it is always infinitely long. Roughly speaking, it's the shape you can draw with a (long, long, long) ruler. A curve, on the other hand, is anything you can draw on a sheet of paper without ever lifting your writing utensil. And—it's in the name—it can be *curved* (not straight).

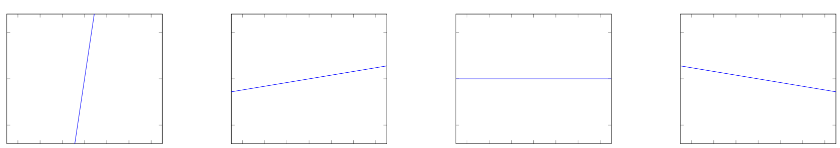


Above, you will see an example of a line on the left, and an example of a curve on the right. *Every line is an example of a curve, but not every curve is a line.*

Here are some facts about lines that you might remember:

1. If you choose two different points on the plane, there is a line that goes through those two points. Moreover, there is exactly *one* line that goes through those two points.

2. Every line has a number called a *slope* associated to it. Informally, slope measures how "tilted" a line is. If the slope is zero, the line is flat. If the slope is negative, the line is tilted downward. If the slope is positive, the line is tilted upward.



   Above, you see pictures of lines of various slope. The leftmost line has a very large, positive slope. (The slope is so large, the line almost looks vertical.) The horizontal line has slope zero. The rightmost line has negative slope.

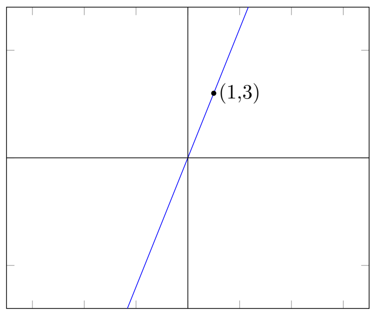
Remark 1.1.1. This is meant to be reminiscent of the usual use of the word "slope" in everyday life. You might have learned from previous classes that "positive slope" is the same thing as "uphill," while "negative slope" is the same thing as "downhill" as you move from left to right.

1.2 Example: Constant velocity

Dorothy is walking at a constant speed of three miles per hour. Let $f(t) = 3t$ denote the function that tells us how far Dorothy has walked (in miles) at time t (in hours).

For example, at $t = 0$, we see that $f(t) = 3 \cdot 0 = 0$, so Dorothy has walked 0 miles. At $t = 1$, $f(1) = 3 \cdot 1 = 3$, so Dorothy has walked 3 miles after 1 hour. At $t = 3$ hours, Dorothy has walked $f(3) = 3 \cdot 3 = 9$ miles.

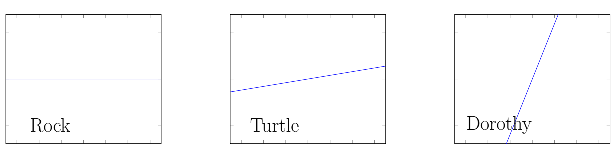
What does the graph of $f(t)$ look like?



Above, you see the graph of $f(t)$ in blue. (The t-axis, also known as the time axis, is horizontal.) For example, the point $(1, 3)$ is on this graph. What you immediately see is that the graph is a line!

In general, it turns out: *If something moves with constant velocity, the position-versus-time graph will always be a straight line.* In the above example, Dorothy moved with constant velocity, so her position-versus-time graph was a straight line.

Here are three position-versus-time graphs of three objects moving at constant velocity: A rock, moving at 0 miles per hour, a turtle, moving at 0.2 miles per hour, and Dorothy, moving at 3 miles per hour.



As you can see, *the faster something is moving, the steeper the line.* Well, we saw above that steep lines have large slopes, so we can conclude

the faster something is moving, the larger the slope.

That is, we witness a relationship between an object's speed, and the slope of its position-versus-time graph.

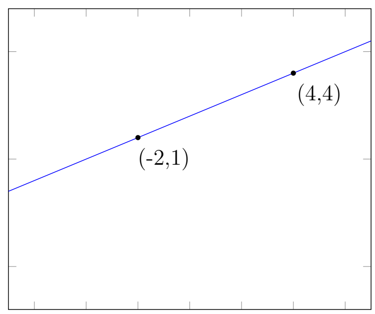
1.3 Calculating slope

Okay, now let's do some math!

The slope of a line is a number. How do we calculate it?

To calculate the slope of a line: Choose two points on the line. Divide the vertical *rise* between the points by the horizontal *run* of the points. You may have seen this in a past math class as "rise over run."

Example 1.3.1. Below is the graph of a line.



The points $(-2, 1)$ and $(4, 4)$ are on this line. What is the slope of this line?

Answer: The "vertical rise" between the two points is the vertical difference between the two points. The point $(4, 4)$ has vertical coordinate 4, while the point $(-2, 1)$ has vertical coordinate 1, so the rise is

$$\text{rise} = 4 - 1 = 3.$$

The "horizontal run" between the two points is the horizontal difference between the two points. The point $(4, 4)$ has horizontal coordinate 4, while the point $(-2, 1)$ has horizontal coordinate -2, so the run is

$$\text{run} = 4 - (-2) = 4 + 2 = 6.$$

Then the slope is given by rise over run, so

$$\frac{\text{rise}}{\text{run}} = \frac{3}{6} = \frac{1}{2}.$$

In other words, the slope of this line is 1/2. (Equivalently, the slope is given in decimals as 0.5.)

Remark 1.3.2. Above, I always measured *from* the point $(4,4)$ to the point $(-2,1)$ when measuring rise and run. You could have measured it the other way: From $(-2,1)$ to $(4,4)$. You'll get the same answer, so long as you are *consistent* about measuring *both* rise and run that way. For example, the rise is given by

$$1-4=-3$$

and the run is given by

$$-2-4=-6.$$

So rise over run is

$$\frac{-3}{-6}=\frac{1}{2}.$$

Tip 1.3.3. By the very definition of slope, you have to be comfortable with division when calculating slopes. This means *you will have to be comfortable with fractions*! If you find at any point that fractions are giving you trouble, don't despair: Just practice, practice, practice with fractions.

1.4 For today's quiz, and for next lecture

1.4.1

Given two points, be prepared to find the slope of the line between those two points.

Sometimes, we will call our two points P and Q.

Example 1.4.1. For each of the following pairs of points P and Q, find the slope of the line passing through P and Q.

1. $P=(1,1)$, $Q=(2,2)$.
2. $P=(1,1)$, $Q=(4,4)$.
3. $P=(1,1)$, $Q=(7,6)$.
4. $P=(1,3)$, $Q=(7,3)$.

The slopes are 1, 1, 6/5, and 0, respectively.

1.4.2

You will also have to know how to do the following. Given a function f and two numbers b and a, you must find the slope of the line between the points $(a,f(a))$ and $(b,f(b))$. You will practice this in lab.

1.4.3

Finally, don't forget to fill out the survey. Hiro has given you the link to the survey either via e-mail or on the course website.

1.5 Lab exercises: Secant lines

Exercise 1.5.1. Let $f(x) = 3x + 5$.

(a) What is $f(2)$?

(b) What is $f(7)$?

(c) Consider the two points $P = (2, f(2))$ and $Q = (7, f(7))$. (If this notation is confusing, you may want to plug in the actual numbers you got for $f(2)$ and $f(7)$ in the previous parts.) Find the slope of the line between these two points.

Exercise 1.5.2. Let $f(x) = x^2 - 3$.

(a) What is $f(2)$?

(b) What is $f(7)$?

(c) Consider the two points $P = (2, f(2))$ and $Q = (7, f(7))$. Find the slope of the line between these two points.

(d) Draw the graph of $f(x)$. You can use a graphing calculator to help you draw if you like. (There are plenty online.) However, I do recommend you draw the graph on a sheet of paper—you'll be doing some drawing in the next few parts.

(e) On the same picture, draw the line passing through P and Q.

(f) Now let Q_1 be the point given by $(5, f(5))$. Draw the line passing through P and Q_1.

(g) Now let Q_2 be the point given by $(4, f(4))$. Draw the line passing through P and Q_2.

(h) Now let Q_3 be the point given by $(3, f(3))$. Draw the line passing through P and Q_3.

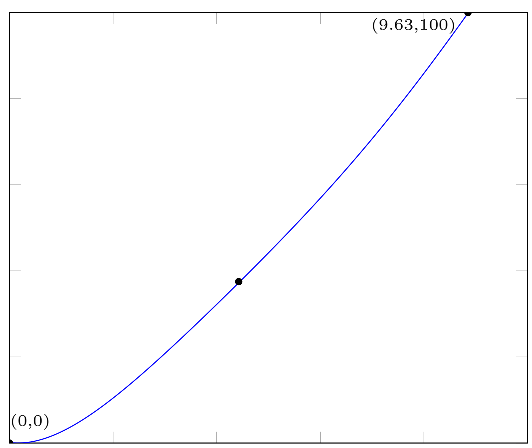
(i) Is there any observation you can make about the lines you've drawn?

Exercise 1.5.3. In the 2012 London Summer Olympics, Usain Bolt ran 100 meters in 9.63 seconds. (A meter is a little more than a yard; roughly, he ran across a football field and then some in 9.63 seconds.) This was—and still is—an Olympic record, and Usain Bolt won a gold medal for his run.

(a) In meters per second, how fast would you say Usain Bolt ran during his run? (You can use a calculator if you like.)

(b) Think about your answer to the previous part of this problem, and discuss with your group: Do you think Usain Bolt was running at that speed for the entirety of those 9.63 seconds? Was he moving slower during certain points of time? Faster?

(c) Below is a position-versus-time graph estimating Usain Bolt's actual run. Can you identify some parts of the graph that capture moments when Bolt was running *slower* than the speed you got in the first part of this problem? *Faster*?



(d) The black dot in the graph has t-coordinate 4.8. Is there a way you might try to estimate Usain Bolt's speed at that time? Put another way: If Usain Bolt had a speedometer on him, what might that speedometer have read at time $t = 4.8$? (This is not an answer that is "known." So feel free to brainstorm and be creative with your group!)