Extra Credit Assignment 2

Due Friday, September 3, 11:59 PM

In class we learned that $(f+g)' = f' + g'$, and that $(af)' = a \cdot f'$. This means

$$(f+g)'(x) = f'(x) + g'(x), \qquad \text{and} \qquad (af)'(x) = af'(x).$$

Another name for this is "derivatives respect addition" and "derivatives respect scaling." (Multiplying by a number a is sometimes called "scaling" by a.)

Let's see *why* these facts are true! For the fact that "derivatives respect addition," let's first remember that when we *add functions*, we mean that $(f+g)(x) = f(x) + g(x)$.[1] So let's see how far we can simplify the difference quotient for the function $f+g$:

$$\begin{aligned}
&\frac{(f+g)(x+h) - (f+g)(x)}{h} && (1)\\
&= \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h} && (2)\\
&= \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}\\
&= \frac{f(x+h) - f(x) + g(x+h) - g(x))}{h}\\
&= \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h}\\
&= \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} && (3)
\end{aligned}$$

Remember, to find out what $(f+g)'(x)$ is, we need to understand what the first expression (in (1)) approaches as h nears zero. The first equality with (2) is true because we're just writing out what it means to evaluate the function $f+g$. (It's how we add function!) The next few strings are being careful about signs, then rearranging the terms we're adding in the numerator, and then (finally) splitting the fraction into the sum of two parts.[2]

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[1]This addition rule for functions is always true, but the addition rule for *input* variables is not. For example, $f(x+h)$ is *not* equal to $f(x) + f(h)$.

[2]Remember that we can always write $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

Well, the above string of equalities is complicated, but it tells us at the of the day that the first expression (1) is *equal* to the last expression 3. So if I want to understand how the first expression behaves as h nears 0, I just need to understand how the last expression behaves as h nears 0.

But that last expression is the sum of two difference quotients! A difference quotient for f, and a difference quotient for h. And we know how each of these behaves as h nears 0—the difference quotient for f becomes $f'(x)$, and the difference quotient for g becomes $g'(x)$.[3] Using symbols, that the two expressions (1) and (3) are equal means that

$$\lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right)$$

and in turn,

$$\begin{aligned} &\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\ &= \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) + \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right) \\ &= (f'(x)) + (g'(x)) \,. \end{aligned} \tag{4}$$

That's a *lot* of equals signs, but tracing through all of them, we finally see

$$(f+g)'(x) = f'(x) + g'(x).$$

That was long, wasn't it? But we learn, in the end, what we were looking for.

Here's the extra credit assignment: Imitating what I just did if you have to, explain to me the following:

(a) Why is it true that

$$(af)'(x) = a \cdot f'(x)?$$

(b) Why is it also true that

$$(f-g)'(x) = f'(x) - g'(x)?$$

[3] This is using the *definition* of $f'(x)$ and $g'(x)$.