Related Rates and Implicit Differentiation Practice Problems for Exam I

1.2

The area of a square of edge length l is given by

$$A = l^2.$$

The square's area is changing with time. If at time t, the area is changing at 3 meters squared per second, how fast is the edge length changing? Does your answer depend on the edge length at time t?

1.3

The area of a rectangle of width w and length l is given by

$$A = wl.$$

At time $t = 5$ seconds, we are told that: $l = 3$ and $w = 1$, and that moreover, l is changing at a rate of 7 units per second and w is changing at a rate of 2 units per second. At what rate is the area changing at $t = 5$ seconds?

1.4

The population of Calculus Town t years from now is modeled by the function

$$P(t) = 13e^{A(t)t}$$

where $A(t)$ is a function measuring affordability, and $P(t)$ is in units of persons. Moreover, based on recent trends, the affordability of the town is predicted to be *decreasing*; in fact, we are told that $A'(1) = -3$. But we are told that the town will remain rather affordable a year from now, so that $A(1) = 5$. Based on this model, at what rate will the population $P(t)$ be changing at $t = 1$?

1.5

Using implicit differentiation, find the slope of the tangent line to a point (x, y) on the shape given by the equation

$$e^y e^x y^2 = x.$$

1.6

Consider the shape given by those points (x, y) satisfying

$$x^4 + y^4 = 17.$$

Find the slope of the tangent line to this shape at the point $(1, 2)$.

Solutions: Related Rates and Implicit Differentiation

1.2

We write the area A and the edge length l as functions of time, so $A(t) = l(t)^2$. Take the derivative of both sides (using the chain rule on the right) to find that

$$A'(t) = 2l(t)l'(t).$$

In other words, if $A'(t) = 3$, we know that $3 = 2l(t)l'(t)$. So we conclude that $l'(t) = \frac{3}{2l(t)}$. In other words, the rate of change of the edge length l is given by $\frac{3}{2l(t)}$. So yes, the answer *does* depend on $l(t)$.

1.3

We write $A(t) = w(t)l(t)$. Using the product rule, we find

$$A'(t) = w'(t)l(t) + w(t)l'(t).$$

We are given that $l(5) = 3, w(5) = 1, l'(5) = 7, w'(5) = 2$. Hence

$$A'(t) = 2 \cdot 3 + 1 \cdot 7 = 13.$$

1.4

Taking the derivative (using the chain rule), we find

$$P'(t) = 13e^{A(t)t} \cdot (A(t)t)'.$$

Using the Leibniz rule on the last term, we find

$$P'(t) = 13e^{A(t)t} \cdot (A'(t)t + A(t)).$$

We are given that $A(1) = 5, A'(1) = -3$, so:

$$P'(1) = 13e^{5\cdot 1} \cdot (-3 \cdot 1 + 5) = 2 \cdot 13 \cdot e^5 = 26e^5.$$

So at $t = 1$, the town's population will still be *increasing*, at a rate of $26e^5$ people per year.

1.5

We find

$$\frac{d}{dx}(e^y e^x y^2) = \frac{d}{dx}(x) \tag{21}$$

$$\frac{d}{dx}(y^2 e^y e^x) = \frac{d}{dx}(x) \tag{22}$$

$$(y^2 e^y)' e^x + (y^2 e^y)(e^x)' = (x)' \tag{23}$$

$$(y^2 e^y)' e^x + (y^2 e^y)(e^x)' = 1 \tag{24}$$

$$(y^2 e^y)' e^x + (y^2 e^y) e^x = 1 \tag{25}$$

$$((y^2)' e^y + y^2 (e^y)') e^x + (y^2 e^y) e^x = 1 \tag{26}$$

$$(2y \cdot y' \cdot e^y + y^2 \cdot e^y \cdot y') e^x + (y^2 e^y) e^x = 1 \tag{27}$$

$$y'(2y \cdot e^y + y^2 \cdot e^y) e^x + (y^2 e^y) e^x = 1 \tag{28}$$

$$y'(2y + y^2) e^y e^x = 1 - y^2 e^y e^x \tag{29}$$

$$y' = \frac{1 - y^2 e^y e^x}{(2y + y^2) e^y e^x} \tag{30}$$